On Entropy and Partition Functions of 4D BPS Black Holes — A Review

Zlatibor — 2004

D=4 Extremal black holes arising in M-theory compactifications on $\underset{\nearrow\ CY_3}{X} \times S^1$ → N=2 supersymm

Black hole microscopics (Maldacena, Strominger Witten, hep-th/9711053)

5-brane wrapped on $\mathcal{P} \times S^1$
↑ four cycle in $X = CY_3$

like D1 brane wrapped on a one-cycle of T^2
$(p^1 = 1,\ p^2 = 5)$

winding numers $p^A \in H_4(X, Z)$ homology class

$\mathcal{P} = p^A \Sigma_A$ $\qquad \Sigma_A$: homology basis

triple intersection $\boxed{C_{ABC}\, p^A p^B p^C}$

in addition second Chern class

two cycle $\in H_2(X, Z)$

choose homology basis Σ^A, such that $\Sigma^A \cdot \Sigma_B = \delta^A_B$

$$C_2 = c_{2A} \Sigma^A$$

$$\int_{\mathcal{P}} c_2(X) = \boxed{c_{2A}\, p^A}$$

↑ 4-form by Poincaré duality

Supersymmetry : $\mathcal{P}$ holomorphic

Relevant degrees of freedom: massless fluctuations of the wrapped cycles

determined by topological data
in certain limits

→ supersymmetric sigma model (2D)
assume momentum along S^1 → q_0

Count (through Cardy's formula for the degeneracy of states for large q_0) yields entropy

$$\frac{S(p,q)}{\pi} = 2\sqrt{\frac{1}{6}\,|\hat{q}_0|\left(C_{ABC}\,p^A p^B p^C + c_{2A}\,p^A\right)}$$

with $\hat{q}_0 = q_0 + \frac{1}{12} D^{AB} q_A q_B$

MSW effect associated with membrane charges

generically $S \sim \sqrt{Q^4}$

c_{2A} subleading.

p magnetic BH charges

q electric BH charges

Macroscopic description

Compactification on $CY_3 \times S^1$ leaves $N=2$ supersymmetry

P winding & S^1 momentum & membrane charges

p^A q_0 q_A

$N=2$ vector multiplets (related to harmonic forms on CY_3)

complex scalars X^I: X^0, X^A $A=1,\dots,n$
define (projectively) special Kähler space of complex dimension n
$\leftrightarrow$ parametrizes the moduli space of CY_3
(special geometry)

gauge fields : W_μ^I: W_μ^0, W_μ^A
$\rightarrow$ charges p^0, p^A q_0, q_A

$N=2$ supergravity yields charged black holes

$\rightarrow$ extremal Reissner-Nordstrom black holes

solitonic solutions with $N=1$ supersymmetry (i.e. BPS) interpolating between two $N=2$ supersymmetric field configurations:

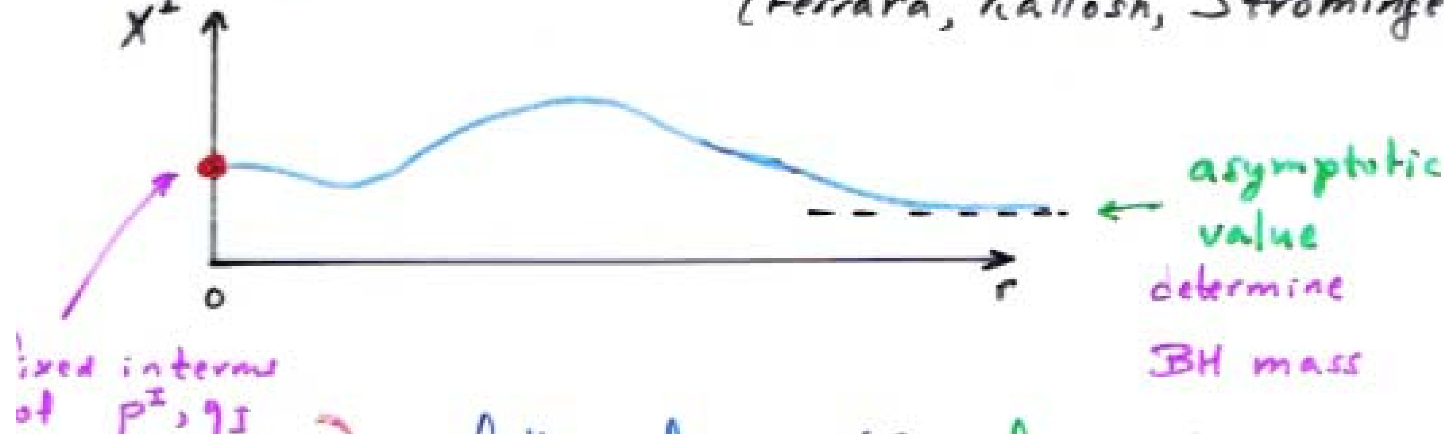
- $r=0$ horizon : Bertotti-Robinson $AdS_2 \times S^2$
- $r=\infty$ spatial inf: flat Minkowski

$N=2$ fully supersymmetric field configurations are very restricted.

For BH with charges p^I, q_I the moduli assume special values at the supersymmetric horizon, determined by the charges.

Fixed point behaviour (attractor mechanism)

(Ferrara, Kallosh, Strominger)



follow from $\langle \delta_{susy} \text{fermion} \rangle = 0$

<u>Note</u> each point of the X^I trajectory defines a corresponding CY_3

flow in CY_3 moduli space (Moore, Denef)

<u>2 more ingredients:</u>

- $N=2$ supergravity Lagrangians are encoded in holomorphic function $F(X)$, homogeneous

$$F(\lambda X) = \lambda^2 F(X)$$

- Electric/magnetic duality

E/M duality (of field equations)

$F_{\mu\nu}{}^{I}$, $G_{\mu\nu I} \propto \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}^{I}}$ rotate by $Sp(2n+2)$

(p^I, q_I) symplectic vector

supersymmetry links $F_{\mu\nu}{}^{I} \leftrightarrow X^I$

preserving this relation

$X^I, F_I = \frac{\partial F}{\partial X^I}$ also symplectic (holomorphic) vector

if there is a relation at the horizon between X^I and p^I, q_I, it must preserve this symplectic covariance

$$\bar{Z}\begin{pmatrix} X^I \\ F_I \end{pmatrix} - Z\begin{pmatrix} \bar{X}^I \\ \bar{F}_I \end{pmatrix} = i\, e^{-\frac{1}{2}K}\begin{pmatrix} p^I \\ q_I \end{pmatrix}$$

with $e^{-K} = i\left(\bar{X}^I F_I - \bar{F}_I X^I\right)$ symplectic $U(1)$ invariant

$$\rightarrow \quad Z = e^{\frac{1}{2}K}\left(p^I F_I - q_I X^I\right)$$

holomorphic BPS mass

for certain $F(X)$ and $q_I\, p^I$ this can be solved in closed form

cf. (Shmakova hep-th/9612076

Behrndt, Cardoso, dW, Kallosh, Lüst, Mohaupt hep-th/9610105) relevant example:

$$F(X) = D_{ABC} \frac{X^A X^B X^C}{X^0}$$

redefine $\quad Y^I = e^{\frac{1}{2}K} \bar{Z} X^I$ $\quad$ projectively invariant

$$F(Y) = D_{ABC} \frac{Y^A Y^B Y^C}{Y^0}, \quad |Z|^2 = p^I F_I - q_I Y^I$$

choose $\boxed{p^0 = 0}$ $\rightarrow$ Y^0 real

$$(Y^0)^2 = \frac{D_{ABC}\, p^A p^B p^C}{4 \hat{q}_0}$$

$$Y^A = \frac{Y^0}{6} D^{AB} q_B + \tfrac{1}{2} i p^A$$

where

$$D_{AB} = D_{ABC}\, p^C \qquad D^{AB} D_{BC} = \delta^A_C$$

$$\hat{q}_0 = q_0 + \tfrac{1}{12} D^{AB} q_A q_B$$

MSW effect

$$ds^2 = -e^{-2f(r)} dt^2 + \underbrace{e^{2f(r)}}_{\to \frac{|Z|^2}{r^2}} \left(dr^2 + r^2 d\Omega_2^2\right)$$

$\Rightarrow$ **Area BH** $= 4\pi |Z|^2$ $\leftarrow$ in Planck units

Bekenstein-Hawking

$$\frac{S}{\pi}(p,q) = |Z|^2 = 2\sqrt{\tfrac{1}{6} |\hat{q}_0|\, C_{ABC}\, p^A p^B p^C}$$

$D_{ABC} = -\frac{1}{6} C_{ABC}$ $\quad$ triple intersection number CY

This coincides with MSW for $c_{2A} = 0$.

MSW:

$$\frac{S}{\pi} = 2\sqrt{\frac{1}{6}|\hat{q}_0|\left(C_{ABC}\,p^A p^B p^C + c_{2A}\,p^A\right)}$$

no longer homogeneous in the charges

suggests:

$$F(X,A) = \frac{D_{ABC}\,X^A X^B X^C + d_A X^A A}{X^0}$$

A ? holomorphic + homogeneous of 2nd degree

so that $F(\lambda X, \lambda^2 A) = \lambda^2 F(X,A)$

- what is A ?
- what value does it take at the horizon?
- can the previous results (fixed point, etc) still be proven?
- but assuming these questions can be answered the MSW result is not reproduced!

(Behrndt et al hep-th/9801081)

R^2-terms

$$\delta\psi_\mu \sim D_\mu \epsilon + T_{\rho\sigma}\, \gamma^{\rho\sigma} \gamma_\mu \epsilon$$

auxilary ~ "graviphoton field strength" (misnomer)

there exists a reduced scalar chiral supermultiplet

$$\hat{A} : T^2 + \theta T \partial\psi + \theta^2 T R + \cdots + \theta^4 R^2$$

(typical terms)

R ~ Riemann tensor

$\propto A$ weight 2

can be incorporated into $F(X) \longrightarrow F(X, A)$

$$F(\lambda X, \lambda^2 A) = \lambda^2 F(X, A)$$

leads to a Lagrangian involving R^2-terms

FURTHERMORE: Bekenstein-Hawking must be modified!
using Wald's prescription based on a conserved surface (Noether) charge which ensures 1st law of BH mechanics:

$$S \propto \left.\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho}{}^{\sigma}}\right|_{\text{horizon}} \sim |Z|^2 + F_A \, T^2$$

$\propto A$

$\mathcal{O}(R)$ terms cancel because of $N=2$ supersymmetry at horizon

Assuming fixed point etc as before, including for $T_{\mu\nu}$ (leads to $A|_{hor} = -64\, \bar{Z}^{-2} e^{-K}$)

$$\frac{S}{\pi}(p,q) = |Z|^2 - 256\ \mathrm{Im}\, F_A$$

area (in presence of R^2)

non area modification

general N=2 formula consistent with EM duality

(Cardoso, dW, Mohaupt, hep-th/9812082
9906094
9910179)

⇒ now MSW entropy is reproduced exactly!

All ingredients have subsequently been established including fixed point behaviour, etc, as well as the interpolating multi-center solutions

(Cardoso, dW, Käppeli, Mohaupt
hepth/0009234)

note fixed point

$A \sim T^2 \sim \bar{Z}^{-2} e^{-K}$ was already valid without R^2-terms, but it was never explicitly needed for the entropy

hypermultiplets

no fixed point behaviour (arbitrary constants)

no contribution to $\frac{S}{\pi}$

chiral invariants

all this based on chiral invariants ("F-terms")

Recently (Ooguri, Strominger, Vafa; hep-th/0405146) the universal $N=2$ entropy formula was related to topological string theory.

define: $$Y^I = \frac{1}{2}\left(\frac{\phi^A}{\pi} + i\, p^A\right)$$

$$\mathcal{F}(p,\phi) = 4\pi \operatorname{Im} F(Y,\Upsilon)\Big|_{\Upsilon=-64}$$

use homogeneity:

$$\mathcal{S}(p,q) = \mathcal{F}(p,\phi) - \phi^I \frac{\partial \mathcal{F}(p,\phi)}{\partial \phi^I}$$

where $$q_I = \frac{\partial \mathcal{F}(p,\phi)}{\partial \phi^I}$$

Φ^I: 'chemical' (electric) potentials for q_I

This is a Legendre transform

$$\rightarrow \quad \mathcal{F}(p,\phi) = \ln Z^{BH}(p,\phi)$$

Z^{BH}: mixed partition function

p: microcanonical
ϕ: canonical

one expects $$Z^{BH}(p,\phi) = \sum_{\{q\}} \Omega(p,q)\, e^{-\phi^I q_I}$$

microcanonical entropy
→ microscopic degeneracies

→ for large q: $S(p,q) \approx \Omega(p,q)$

For case at hand:

$$\mathcal{F}(p,\phi) = \frac{\pi^2}{\phi^0}\Big\{-D_{ABC}\,p^A p^B p^C + 4\, d_A\, p^A\, \Upsilon + \frac{3}{\pi^2} D_{AB}(p)\,\phi^A\phi^B\Big\}$$

note:

$$Z^{BH}(p,\phi) = e^{\mathcal{F}} = \left| e^{-2\pi i F}\right|^2$$

$$e^{-2\pi i F}\Big|_{\Upsilon = -256} = Z^{top} \quad \text{topological string.}$$

indeed

$$-2\pi i\, F(Y) = -\frac{(2\pi)^3 i}{g_{top}^2} D_{ABC}\, t^A t^B t^C - \tfrac{1}{2} i\pi\, d_A\, t^A\, \Upsilon$$

with $t^A = \frac{Y^A}{Y^0}$ special / flat coordinates

$$(g_{top})^2 = \left(\frac{2\pi}{Y^0}\right)^2$$

the topological partition function of a twisted sigma model on CY_3 target space, summed over Riemann surfaces of genus g

$$\ln Z^{top} \sim \sum_{g=0}^{\infty} F^{(g)}(Y)\,\Upsilon^g$$

$$\sim \sum_{g=0}^{\infty} (g_{top})^{2g-2} F^{(g)}(t)\,\Upsilon^g$$

How to test this relationship further?

Coincidence?

to rewrite the entropy as $\mathcal{F} - \phi\cdot\frac{\partial}{\partial\phi}\mathcal{F}$

hardly any of the factors seems important .....

on the microscopic side, several directions for future research indicated by OSV

on the macroscopic side:

determine the partition function from the Euclidean description with temperature β^{-1}

for Schwarzschild BH: action vanishes and thermodynamics arises from the surface term

<u>here</u> action does not vanish. There is one term that does <u>not</u> vanish on the horizon

$$\mathcal{L} = \frac{1}{16}\,\mathrm{Im}\left(F(X,A)\,\bar{A}\right)$$

In any case, to work out the partition function as a function of p's and ϕ's is in principle not obvious

Another way is to compare terms in the effective action (from string amplitudes?) directly with the topological string (cf Antoniadis, Gava, Narain, Taylor, 1995)