Conformal and modular symmetry in four dimension

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2D - math. background:

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Symmetry combined with the QFT principles

Mathematicians involved

John Wallis 1616 - 1703 Elliptic integrals

Jacob Bernoulli 1654 - 1705

Leonard Euler 1703 - 1783

Adrien-Marie Legendre 1752 - 1833

Carl Friedrich Gauss 1777 - 1855 1798

Niels Henrik Abel 1802 - 1829 1825 elliptic functions

Carl Gustav Jacobi 1804 - 1851 1828

Joseph Liouville 1809 - 1882

Ferdinand Gotthold Eisenstein 1823 - 1852 $G_{2k}(\tau)$

Karl Weierstrass 1815 - 1897 $\wp(z,\tau)$

Richard Dedekind 1831 - 1916 $\eta(\tau)$

Modular forms

$$G_{2k}(\tau) = \frac{(2k-1)!}{2(2\pi i)^{2k}} \sum_{(m,n)\neq(0,0)} (m\tau+n)^{-2k}$$

$$= -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \frac{n^{2k-1}}{1-q^n} q^n, \quad q = e^{2\pi i \tau}$$

$\operatorname{Im}\tau > 0$ $|q|<1$

Modular invariant:

$$j(\tau) = \frac{[240\, G_4(\tau)]^3}{[\eta(\tau)]^{24}} = \frac{1}{q} + 744 + 196884q + \cdots$$

c-3

2 Conformal QFT on compactified space-time $\bar{M}$

$$\bar{M} := \{ z \in \mathbb{C}^4 ; \ \frac{z}{R} = e^{2\pi i \zeta} u, \ \ u = \{u_\alpha, \alpha = 1, \dots, 4\}, u^2 : \underline{u}^2 + u_4^2 = 1 \}$$

$$\bar{M} = \mathbb{S}^3 \times \mathbb{S}^1 / \mathbb{Z}_2 \qquad \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{1, -1\}.$$

Universal cover $\mathbb{R} \times \mathbb{S}^3$ Einstein static Universe (I. Segal)

$$g_c : M (\ni x) \to \bar{M} (\ni z) \quad g_c x = z \quad \underline{z} = \frac{\underline{x}}{2\omega}, \ z_4 = \frac{R - \frac{x^2}{4R}}{2\omega}$$

$$2\omega \equiv 2\omega\left(\frac{x}{2R}\right) = 1 + \frac{x^2}{4R^2} - i\frac{x^0}{R} \qquad dz^2 = \frac{dx^2}{4\omega^2}$$

$$\underline{z}(x, R) = \underline{x} + O\left(\frac{|x|^2}{R}\right), \quad z_4(x, R) - R = i x^0 + O\left(\frac{|x|^2}{R}\right)$$

$|x|^2 := \underline{x}^2 + x_0^2 \qquad \underline{x}^2 = x_1^2 + x_2^2 + x_3^2$ ($z_\alpha - R\vec{e}_4$ approximate Euclidean x (i.e. $\underline{x}, ix^0$).

$\phi(x + iy) |0\rangle$ analytic in the tube domain

$$\mathcal{T}_+ = \{ x + iy ; \ x \in \mathbb{R}^4, \ y^0 > |\underline{y}| \ (y \in \mathbb{R}^4) \}$$

$$g_c \mathcal{T}_+ = T_+ := \{ z \in \mathbb{C}^4 ; \ |z|^2 := \sum_{\alpha=1}^{4} |z_\alpha|^2 < \frac{1}{2}\left(R^2 + \frac{|z^2|^2}{R^2}\right), \ |z^2| < R^2 \}$$

3. z-picture fields: formal power series

$$\phi(z) = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{\infty} (z^2)^n \phi_{\{n,m\}}(z), \quad \Delta_4 \phi_{\{n,m\}}(z) = 0$$ - homogeneous harmonic polynomials of degree m.

$$(z_{12}^2)^N \left(\phi(z_1) \phi^*(z_2) - (-1)^{2d} \phi^*(z_2) \phi(z_1) \right) = 0 \quad d \in \tfrac{1}{2}\mathbb{N} \text{ "dimension"}$$

strong locality

Same for 2 different fields

Θ-functions

Riemann: $\Theta(z,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n z}$, $q^{\frac{1}{2}} = e^{i\pi\tau}$

Heisenberg-Weyl group:

$$(U_a f)(z) = e^{i\pi(a^2\tau + 2az)} f(z + a\tau), \quad U_{a_1} U_{a_2} = U_{a_1+a_2}$$

$$(V_b f)(z) = f(z+b), \quad V_b U_a = e^{2\pi i a b} U_a V_b$$

commute for integer a, b

Jacobi Θ-functions (Mumford: Tata Lectures on Theta)

$$\Theta_{\mu\nu}(z,\tau) = V_{\frac{\nu}{2}} U_{\frac{\mu}{2}} \Theta(z,\tau), \quad \mu,\nu = 0,1$$

$$= e^{i\frac{\pi\mu}{2}\left(\frac{\mu}{2}\tau + 2z + \nu\right)} \Theta\left(z + \frac{\mu\tau+\nu}{2}, \tau\right)$$

$\Theta_{00} = \Theta$ Twisted periodicity

$$\Theta_{\mu\nu}(z+1,\tau) = (-1)^{\mu}\, \Theta_{\mu\nu}(z,\tau)$$

$$\Theta_{\mu\nu}(z+\tau,\tau) = (-1)^{\nu} q^{\frac{\mu-1}{2}} e^{-2\pi i z}\, \Theta_{\mu\nu}(z,\tau)$$

Schrödinger equation:

$$i\frac{\partial}{\partial\tau}\Theta_{\mu\nu}(z,\tau) = \frac{1}{4\pi}\frac{\partial^2}{\partial z^2}\Theta_{\mu\nu}(z,\tau)$$

J. Fourier "heat" equation

Modular propertis of Eisenstein series

$G_{2k}(\tau)\, d\tau^k$ is a modular form for $k = 2, 3, \ldots$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$: $\frac{1}{(c\tau+d)^{2k}} G_{2k}(\gamma\tau) = G_{2k}(\tau)$

but $\qquad \gamma\tau = \frac{a\tau+b}{c\tau+d}$

$$(c\tau+d)^{-2} G_2(\gamma\tau) = G_2(\tau) + \frac{i}{4\pi}\frac{c}{c\tau+d}$$

$$G_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} n^{2k-1}\frac{q^n}{1-q^n}$$

$G_2^*(\tau) = G_2(\tau) + \frac{1}{8\pi \operatorname{Im}\tau}$ is modular

Fermionic mean energy

$$F(\tau) := 2\,G_2(\tau) - G_2\left(\frac{\tau+1}{2}\right)$$

$\Gamma_\Theta \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Energy distribution for free fields

Partition function: $Z(\tau) = tr_{\mathcal{V}}\, q^H = \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}})^{d_f(n)}}{(1-q^n)^{d_B(n)}}$

$d_B(n)$ dimension of $\left.\begin{array}{l}\text{boson}\\ \text{fermion}\end{array}\right\}$ 1-particle state of energy $\left\{\begin{array}{l} n \\ n-\frac{1}{2}\end{array}\right.$
f

For a free scalar field $\varphi(z) = \sum_{n=1}^{\infty} \{\varphi_{-n}(z) + \frac{1}{z^{2n}} \varphi_n(z)\}$

$\Delta_4 \varphi(z) = 0$

$\varphi_{\pm n}(z)$ are homogeneous harmonic polynomials of degree $n-1$. $\quad d_\varphi(n) = \binom{n+2}{3} - \binom{n}{3} = n^2$

$\binom{n+2}{3}$ = dimension of the space of all homogeneous polynomials of a 4-vector z of degree $n-1$

$$\langle H \rangle_q = \frac{1}{Z(\tau)} tr_{\mathcal{V}}(H q^H) = \frac{1}{Z(\tau)} q \frac{\partial}{\partial q} Z(\tau)$$

$$H \to \frac{\hbar c}{R} H \Leftrightarrow \tau \to \frac{\hbar c}{R} \tau \qquad H_R = P^0 + \frac{1}{4R^2} K^0$$

$=: H_R$

$\mathcal{V}_R = 2\pi^2 R^3$ = volume of 3-space

$$\mathcal{E}(\beta,R) := \frac{\langle H_R + E_R \rangle_q}{\mathcal{V}_R} \underset{q = e^{-\beta}}{=} \frac{\hbar c}{R \mathcal{V}_R} G_4\left(i \frac{\hbar c}{2\pi R}\beta\right) = \frac{1}{\mathcal{V}_R} \sum_{n=1}^{\infty} \frac{n^2 h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} + \frac{E_R}{\mathcal{V}_R}$$

Planck's formula for $\nu = n \frac{c}{R}$

$G_4(\tau) = \frac{1}{\tau^4} G_4\left(\frac{-1}{\tau}\right)$ for $\tau = i \frac{\hbar c \beta}{2\pi R}$

$$\mathcal{E}(\beta,R) = \frac{\pi^2}{30 \hbar^3 c^3 \beta^4}\left(1 + O\left(e^{-4\pi^2 \frac{R}{\beta}}\right)\right)$$

Stefan-Boltzmann law