Dilaton field contribution to non-commutativity

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Known result

- Dp-brane world-volume is non-commutative if open string ends on Dp-brane with Neveu-Schwarz field $B_{\mu\nu}$

New result

(Sazdović, hep-th/0408131)

- The dilaton field Φ turns the coordinate in $\partial_\mu \Phi$ direction to **commutative** one

Method

- We apply canonical method
- We treat boundary conditions as primary constraints

Definition of the model

Action

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu} + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} \mathcal{F}_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi R^{(2)} \right\}$$

- – $x^\mu(\xi)$ $(\mu = 0, 1, ..., D-1)$ space-time coordinates
  – ξ^α $(\alpha = 0, 1)$ world-sheet coordinates
  – $x^i(\xi)$ $(i = 0, 1, ..., p)$ Dp-brane coordinates

- Open string propagates in x^μ dependent background
  – metric tensor $G_{\mu\nu}$
  – dilaton field Φ
  – antisymmetric tensor field $B_{\mu\nu} = -B_{\nu\mu}$
  – $U(1)$ gauge field, living on Dp-brane A_i
  – modified Born-Infeld field strength

$$\mathcal{F}_{\mu\nu} = B_{\mu\nu} + (\partial_i A_j - \partial_j A_i)\delta^i_\mu \delta^j_\nu$$

- Conformal gauge, $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta} \Longrightarrow R^{(2)} = 2\Delta F$

$$S = \kappa \int_\Sigma d^2\xi \left\{ \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu} + \varepsilon^{\alpha\beta}\mathcal{F}_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + 2\Phi e^{2F} \Delta F \right\}$$

Action depends on $F \longrightarrow$ no conformal invariance

Solution of the space-time field equations

- Conditions on world-sheet conformal invariance

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4}\mathcal{F}_{\mu\rho\sigma}\mathcal{F}_{\mu}{}^{\rho\sigma} + 2D_\mu a_\nu = 0$$

$$\beta^{\mathcal{F}}_{\mu\nu} \equiv D_\rho \mathcal{F}^\rho_{\mu\nu} - 2a_\rho \mathcal{F}^\rho_{\mu\nu} = 0$$

$$\beta^\Phi \equiv 4\pi\kappa\frac{D-26}{3} - R + \tfrac{1}{12}\mathcal{F}_{\mu\rho\sigma}\mathcal{F}^{\mu\rho\sigma} - 4D_\mu a^\mu + 4a^2 = 0$$

$a_\mu = \partial_\mu \Phi$, $\mathcal{F}_{\mu\rho\sigma}$ is field strength of the field $\mathcal{F}_{\mu\nu}$

- Exact solution

$$G_{\mu\nu}(x) = G_{\mu\nu} = const\,, \quad \mathcal{F}_{\mu\nu}(x) = \mathcal{F}_{\mu\nu} = const$$
$$\Phi(x) = \Phi_0 + a_\mu x^\mu\,, \quad (a_\mu = const)$$

for $a^2 = \kappa\pi\frac{26-D}{3}$

- Chose for simplicity
  - $\mathcal{F}_{\mu\nu} \to \mathcal{F}_{ij}$ and $a_\mu \to a_i$
  - $G_{\mu\nu} = 0$ for $\mu = i \in \{0, 1, .., p\}$, $\nu = a \in \{p+1, ..., D-1\}$

Canonical analyzes

Sazdović, hep-th/0304085

- – Canonical variables x^μ, π_μ, F, π
  – Canonical Hamiltonian $\mathcal{H}_c = T_- - T_+$
  – Energy momentum tensor components

$$T_\pm = \mp \frac{1}{4\kappa}\left(G^{ij} J_{\pm i} J_{\pm j} + \frac{j}{a^2} i^\Phi_\pm \right) + \frac{1}{2}(i^{\Phi\,\prime}_\pm - F' i^\Phi_\pm)$$

  – Virasoro algebras

$$\{T_\pm, T_\pm\} = -[T_\pm(\sigma) + T_\pm(\bar\sigma)]\delta', \qquad \{T_\pm, T_\mp\} = 0$$

- Currents on the Dp-brane

$$J^i_\pm = P^{T\,ij} j_{\pm j} + \frac{a^i}{2a^2} i^\Phi_\pm, \qquad i^\Phi_\pm = \pi \pm 2\kappa a_i x^{i\prime}$$

$$j_{\pm i} = \pi_i + 2\kappa \Pi_{\mp ij} x^{j\prime}, \qquad j = a^i j_{\pm i} - \frac{1}{2} i^\Phi_\pm$$

$$i^F_\pm = \frac{a^i}{a^2} j_{\pm i} - \frac{1}{2a^2} i^\Phi_\pm \pm 2\kappa F',$$

$$\left(\Pi_{\pm ij} = \mathcal{F}_{ij} \pm \frac{1}{2} G_{ij} \right)$$

  – Projection operators

$$P^L_{ij} = \frac{a_i a_j}{a^2}, \qquad P^T_{ij} = G_{ij} - \frac{a_i a_j}{a^2}$$

Open string boundary conditions

- Boundary conditions

$$\left(\frac{\partial S}{\partial x'^\mu} \delta x^\mu + \frac{\partial S}{\partial F'} \delta F \right) |_{\partial\Sigma} = 0$$

- x^i, F $\longrightarrow$ Neumann boundary conditions
  arbitrary variations δx^i and δF on the string endpoints

$$\gamma_i^{(0)}|_{\partial\Sigma} = 0 \,, \qquad \gamma^{(0)}|_{\partial\Sigma} = 0$$

$$\gamma_i^{(0)} \equiv \frac{\delta S}{\delta x'^i} = \kappa(-G_{ij} x'^j + 2\mathcal{F}_{ij} \dot{x}^j - 2a_i F')$$

$$\gamma^{(0)} \equiv \frac{\delta S}{\delta F'} = -2\kappa a_i x'^i$$

- x^a $\longrightarrow$ Dirichlet boundary conditions
  fixed edges of the string $\qquad \delta x^a|_{\partial\Sigma} = 0$

- In terms of the currents

$$\gamma_i^{(0)} = \gamma_{i-} + \gamma_{i+} \,, \qquad \gamma_{i\pm} \equiv \Pi_{\pm ij} J_\pm^j \mp \frac{a_i}{2} i_\pm^F$$

$$\gamma^{(0)} = \gamma_- + \gamma_+ \,, \qquad \gamma_\pm = \mp \frac{1}{2} i_\pm^F$$

Constraints

- Consider $\gamma_i^{(0)}|_{\partial\Sigma}$ and $\gamma^{(0)}|_{\partial\Sigma}$ as primary constraints

- Using Poisson brackets

$$\{H_c, J_{\pm A}\} = \mp J'_{\pm A}, \quad J_{\pm A} = \{J_{\pm i},\, i^F_\pm,\, i^\Phi_\pm,\, \gamma_{\pm i},\, \gamma_\pm\}$$

  – Diarc consistency conditions generate two infinity sets $\gamma_i^{(n)}|_{\partial\Sigma} = 0$ and $\gamma^{(n)}|_{\partial\Sigma} = 0$, $(n \geq 1)$,

$$\gamma_i^{(n)} \equiv \{H_c, \gamma_i^{(n-1)}\} = \partial_\sigma^n \{\gamma_{i-} + (-1)^n \gamma_{i+}\}$$

$$\gamma^{(n)} \equiv \{H_c, \gamma^{(n-1)}\} = \partial_\sigma^n [\gamma_- + (-1)^n \gamma_+]$$

  – In compact form

$$\Gamma_i(\sigma) \equiv \sum_{n\geq 0} \frac{\sigma^n}{n!}\gamma_i^{(n)}(0) = \gamma_{i-}(\sigma) + \gamma_{i+}(-\sigma)$$

$$\Gamma(\sigma) \equiv \sum_{n\geq 0} \frac{\sigma^n}{n!}\gamma^{(n)}(0) = \gamma_-(\sigma) + \gamma_+(-\sigma)$$

  – On the other endpoint

$$\bar\Gamma_i(\sigma) \equiv \sum_{n\geq 0} \frac{(\sigma-\pi)^n}{n!}\gamma_i^{(n)}(\pi) = \gamma_{i-}(\sigma) + \gamma_{i+}(2\pi - \sigma)$$

$$\bar\Gamma(\sigma) \equiv \sum_{n\geq 0} \frac{(\sigma-\pi)^n}{n!}\gamma^{(n)}(\pi) = \gamma_-(\sigma) + \gamma_+(2\pi - \sigma)$$

- Periodicity
  - Only arguments of positive chirality currents differ
    $\Rightarrow \gamma_{i+}(-\sigma) = \gamma_{i+}(2\pi - \sigma) \qquad \gamma_+(-\sigma) = \gamma_+(2\pi - \sigma)$
  - All currents and all variables x^i, π_i, F and π are periodic in σ: $\quad \sigma \to \sigma + 2\pi$

- All constraints weakly commute with hamiltonian

$$\{H_c, \Gamma_i(\sigma)\} = \Gamma_i'(\sigma)\,, \qquad \{H_c, \Gamma(\sigma)\} = \Gamma'(\sigma)$$

there are no more constraints

- Algebra of constraints

$$\{\Gamma_i(\sigma), \Gamma_j(\bar\sigma)\} = -\kappa \tilde{G}_{ij}\delta'(\sigma - \bar\sigma), \quad \{\Gamma(\sigma), \Gamma(\bar\sigma)\} = 0$$

$$\{\Gamma_i(\sigma), \Gamma(\bar\sigma)\} = -2\kappa a_i \delta'(\sigma - \bar\sigma)\,,$$

were we introduced **effective metric tensor**

$$\tilde{G}_{ij} \equiv G_{ij} - 4\mathcal{F}_{ik}P^{Tkq}\mathcal{F}_{qj}$$

  - We will refer to it as the **open string metric tensor**, the metric tensor seen by the open string

- Conventions: $\quad \tilde{G}^{ij} \quad$ inverse of $\quad \tilde{G}_{ij}$

$$\tilde{V}^i = \tilde{G}^{ij}V_j \qquad \tilde{V}^2 = \tilde{G}^{ij}V_iV_j$$
$$V^i = G^{ij}V_j \qquad V^2 = G^{ij}V_iV_j$$

- For $\bar{a}^2 \neq 0$ all constraints are of the second class

Solution of the boundary conditions

- Introduce Dirac brackets or solve constraints
- Open string variables $q^i,\, p_i,\, f,\, p \qquad \bar{q}^i,\, \bar{p}_i,\, \bar{f},\, \bar{p}$

$$
\begin{aligned}
q^i(\sigma) &= \frac{1}{2}\left[x^i(\sigma) + x^i(-\sigma)\right], \quad & \bar{q}^i(\sigma) &= \frac{1}{2}\left[x^i(\sigma) - x^i(-\sigma)\right] \\
p_i(\sigma) &= \frac{1}{2}\left[\pi_i(\sigma) + \pi_i(-\sigma)\right], \quad & \bar{p}_i(\sigma) &= \frac{1}{2}\left[\pi_i(\sigma) - \pi_i(-\sigma)\right] \\
f(\sigma) &= \frac{1}{2}\left[F(\sigma) + F(-\sigma)\right], \quad & \bar{f}(\sigma) &= \frac{1}{2}\left[F(\sigma) - F(-\sigma)\right] \\
p(\sigma) &= \frac{1}{2}\left[\pi(\sigma) + \pi(-\sigma)\right], \quad & \bar{p}(\sigma) &= \frac{1}{2}\left[\pi(\sigma) - \pi(-\sigma)\right]
\end{aligned}
$$

- The constraints in terms of open string variables

$$\Gamma_i(\sigma) = 2(\mathcal{F}P^T)_i{}^j p_j + \bar{p}_i + \frac{1}{a^2}\mathcal{F}_{ij}a^j p - \kappa\tilde{G}_{ij}\bar{q}^{j\prime} - 2\kappa a_i \bar{f}^{\prime}$$

$$\Gamma(\sigma) = \bar{p} - 2\kappa a_i \bar{q}^{i\prime}$$

  - Symmetric and antisymmetric parts separately vanish
  - Antisymmetric (bar) variables in terms of symmetric ones

$$\bar{p}_i = 0\,, \qquad \bar{q}^{i\prime} = \frac{2}{\kappa}(\tilde{P}^T\mathcal{F})^{ij}\left(P^T{}_j{}^k p_k + \frac{a_j}{2a^2}p\right)$$

$$\bar{p} = 0\,, \qquad \bar{f}^{\prime} = \frac{(\tilde{a}\mathcal{F})^i}{\kappa\tilde{a}^2}p_i$$

Effective theory

- The original variables in terms of new ones

$$x^i = q^i + \frac{2}{\kappa}(\tilde{P}^T \mathcal{F})^{ij} \int^\sigma d\sigma_1 \left(P^T{}_j{}^k p_k + \frac{a_j}{2a^2} p \right), \qquad \pi_i = p_i$$

$$F = f + \frac{(\tilde{a}\mathcal{F})^i}{\kappa \tilde{a}^2} \int^\sigma d\sigma_1 \, p_i \,, \qquad \pi = p$$

- Effective energy-momentum tensor in terms of new variables

$$T_\pm[x^i(q^i, p_i, p), \pi_i(p_i), F(f, p_i), \pi(p)] = \tilde{T}_\pm(q^i, p_i, f, p)$$

  – has exactly the same form as $T_\pm$

$$\tilde{T}_\pm = \mp \frac{1}{4\kappa} \left(\tilde{G}^{ij} \tilde{J}_{\pm i} \tilde{J}_{\pm j} + \frac{\tilde{j}}{\tilde{a}^2} \tilde{i}^\Phi_\pm \right) + \frac{1}{2} (\tilde{i}^{\Phi\prime}_\pm - f' \tilde{i}^\Phi_\pm)$$

  – but in **new background**

$$G_{ij} \to \tilde{G}_{ij} = G_{ij} - 4\mathcal{F}_{ik} P^{Tkq} \mathcal{F}_{qj} \,,$$

$$\mathcal{F}_{ij} \to \tilde{\mathcal{F}}_{ij} = 0 \,, \qquad \Phi \to \tilde{\Phi} = \Phi_0 + a_i q^i$$

Non-commutativity in presence of dilaton

- Poisson brackets between the coordinates
  Separate the center of mass, $x^i_{cm} = \frac{1}{\pi}\int_0^\pi d\sigma x^i(\sigma)$
  $x^i(\sigma) = x^i_{cm} + X^i(\sigma)$

$$\{X^i(\sigma), X^j(\bar\sigma)\} = \Theta^{ij} \begin{cases} -1 & \sigma = 0 = \bar\sigma \\ 1 & \sigma = \pi = \bar\sigma \\ 0 & \text{otherwise} \end{cases}$$

$$\Theta^{ij} = \frac{-1}{\kappa}\tilde P^{Tik}\mathcal{F}_{kq}P^{Tqj} \qquad (\Theta^{ij} = -\Theta^{ji})$$

- Geometrical meaning: induced metrics on M_{D-1} defined by the condition $\Phi(x) = const$

$$P^T{}_{ij} = G^{(D-1)}_{ij}, \qquad \tilde P^T{}_{ij} = \tilde G^{(D-1)}_{ij}$$

$$\Theta^{ij} = \frac{-1}{\kappa}\tilde G^{ik}_{(D-1)}\mathcal{F}_{kq}G^{qj}_{(D-1)}, \quad \tilde G_{ij} = (G - 4\mathcal{F}G^{-1}_{D-1}\mathcal{F})_{ij}$$

- Similar as in the absence of dilaton field

$$\theta^{ij} = \frac{-1}{\kappa}G^{ik}_{eff}\mathcal{F}_{kq}G^{qj}, \qquad G^{ij}_{eff} = (G - 4\mathcal{F}G^{-1}\mathcal{F})^{-1ij}$$

- Conformal part of metric F commutes with x^i

$$\{F(\sigma), x^i(\bar\sigma)\} = 0$$

Commutative Dp-brane direction

- Component $x \equiv a_i x^i$ commutes with all other coordinates

$$a_i P^{Tij} = 0 \quad \Rightarrow \quad a_i \Theta^{ij} = 0$$

$$\{x(\sigma), x^j(\bar{\sigma})\} = 0$$

- The presence of dilaton field linear in coordinates decreases the rank of the antisymmetric matrix Θ^{ij} by 1

  It turns one coordinate, in a_i direction, to **commutative** one

- Symmetric and antisymmetric string parameters
  - Closed string

$$\mathcal{F}^{ij} \pm \tfrac{1}{2} G^{ij} = (G^{-1} \Pi_{\pm} G^{-1})^{ij} \,.$$

  - Open string: dilaton free case

$$\kappa \theta^{ij} \pm \tfrac{1}{2} G^{ij}_{eff} = (G^{-1} \Pi_{\pm} G^{-1}_{eff})^{ij}$$

  - Open string with dilaton (the case of the present paper)

$$\kappa \Theta^{ij} \pm \tfrac{1}{2} \tilde{G}^{ij}_{D-1} = (G^{-1}_{D-1} \Pi_{\pm} \tilde{G}^{-1}_{D-1})^{ij}$$