QFT Lectures on AdS-CFT

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Introduction

We reserve the term "AdS-CFT correspondence" for the field theoretical model that was given by Witten [20] and Polyakov et al. [14] to capture some essential features of Maldacena's Conjecture [16]. It provides the generating functional for conformally invariant Schwinger functions in D-dimensional Minkowski space by using a classical action $I[\phi^{\rm AdS}]$ of a field on $D+1$-dimensional Anti-deSitter space. In contrast to Maldacena's Conjecture which involves String theory, gravity, and supersymmetric large N gauge theory, the AdS-CFT correspondence involves only ordinary quantum field theory (QFT), and should be thoroughly understandable in corresponding terms.

In these lectures, we want to place AdS-CFT into the general context of QFT. We are not so much interested in the many implications of AdS-CFT, than rather in the question "how AdS-CFT works". We shall discuss in particular

- why the AdS-CFT correspondence constitutes a challenge for orthodox QFT
- how it can indeed be (at least formally) reconciled with the general requirements of QFT
- how the corresponding (re)interpretation of the AdS-CFT correspondence matches with other, more conservative, connec-

tions between QFT on AdS and conformal QFT, which have been established rigorously.

The lectures are meant to be introductory. When we refer to rigorous methods and results in QFT, our exposition never has the ambition of being rigorous itself. We shall avoid all technical details, but only point out some of the features which are crucial for some arguments but often enough neglected.

To prepare the ground, we shall in the first lecture remind the reader of some general facts about QFT (and its formal Euclidean functional integral approach), with special emphasis on the passage between real-time QFT and Euclidean QFT, and the positivity properties which are necessary for the probability interpretation of quantum theory.

Only in the second lecture, we turn to AdS-CFT, pointing out its apparent conflict (at a formal level) with positivity. We resolve this conflict by (equally formally) relating the conformal quantum field defined by AdS-CFT with a limit of "conventional" quantum fields which do fulfill positivity.

The third lecture is again devoted to rigorous methods of QFT, which become applicable to AdS-CFT by virtue of the result of the second lecture, and which concern both the passage from AdS to CFT and the converse passage.

To keep the exposition simple, and in order to emphasize the extent to which the AdS-CFT correspondence can be regarded as a model-independent connection, we shall confine ourselves to bosonic (mostly scalar) fields (with arbitrary polynomial couplings), and never mention the vital characteristic problems pertinent to gauge (or gravity) theories.

1 First lecture: QFT

A fully satisfactory (mathematically rigorous) QFT must fulfill a number of requirements. These are, in brief:

- Positive definiteness of the Hilbert space inner product.
- Local commutativity of the fields[1] ϕ_q at spacelike separation.
- A unitary representation of the Poincaré group, implementing covariant transformations of the fields.
- Positivity of the energy spectrum in one, and hence every inertial frame.
- Existence (and uniqueness) of the ground state = vacuum Ω.

Clearly, for one reason or another, one may be forced to relax one or the other of these requirements, but there should be good physical motivation to do so, and sufficient mathematical structure to ensure a safe physical interpretation of the theory. E.g., one might relax the locality requirement at very short distances where it has not been tested directly, as long as macrocausality is maintained; or one might admit modifications of the relativistic energy-momentum relation at very high energies. But it is known that there are very narrow limitations on such scenarios. Hilbert space positivity may be absent at intermediate steps, notably in covariant approaches to gauge theory, but it is indispensable if one wants to saveguard the probabilistic interpretation of expectation values of *observables*.

The above features are reflected in the properties of the vacuum expectation values of field products

$$W(x_1, \dots, x_n) = (\Omega, \phi_q(x_1) \dots \phi_q(x_n)\Omega), \tag{1.1}$$

considered as "functions" (in fact, distributions) of the field coordinates x_i, known as the Wightman distributions.

Local commutativity and covariance appear as obvious symmetry properties under permutations (provided x_i and x_{i+1} are at spacelike distance) and Poincaré transformations, respectively. The uniqueness of the vacuum is a cluster property (= decay behaviour at large spacelike separations). Further consequences for the Wightman distributions will be described in the sequel.

[1] We use the notation ϕ_q in order to distinguish the real-time *quantum* field (an operator[-valued distribution] on the Hilbert space) from the Euclidean field ϕ_E (a random variable) and its representation by a functional integral with integration variable ϕ, see below.

1.1 The Wick rotation

The properties of Wightman functions allow for the passage to Euclidean "correlation functions", known as the "Wick rotation". Because this passage and the existence of its inverse justify the most popular Euclidean approaches to QFT, let us study in more detail what enters into it.

The first step is to observe that by the spectrum condition, the Wightman distributions can be analytically continued to complex points $z_i = x_i + iy_i$ by replacing the factors $\mathrm{e}^{-ik_i \cdot x_i}$ in the Fourier representation by $\mathrm{e}^{-ik_i \cdot z_i}$, provided $z_i - z_{i+1}$ have future timelike imaginary parts (the "forward tube"). The analytically continued distributions are in fact analytic *functions* in the forward tube. The reason is that the momenta $k_i + \ldots + k_{n-1} + k_n$ (being eigenvalues of the momentum operator) can only take values in the future light-cone, so that $\prod_i \mathrm{e}^{-ik_i \cdot z_i} = \mathrm{e}^{ik_n \cdot (z_{n-1} - z_n)} \cdot \mathrm{e}^{i(k_{n-1}+k_n)\cdot(z_{n-2}-z_{n-1})} \cdot \mathrm{e}^{i(k_{n-2}+k_{n-1}+k_n)\cdot(z_{n-3}-z_{n-2})} \cdot \ldots$ decay rapidly if the imaginary parts of $z_i - z_{i-1}$ are future timelike, and otherwise would diverge rapidly for some of the contributing momenta. The Wightman *distributions* are thus boundary values (as $\mathrm{Im}\,(z_i - z_{i+1}) \searrow 0$ from the future timelike directions) of analytic Wightman *functions*.

Together with covariance which implies invariance under the *complex* Lorentz group, the analytic Wightman functions can be extended to a much larger complex region, the "extended domain". Unlike the forward tube, the extended domain contains real points which are spacelike to each other, hence by locality, the Wightman functions are symmetric functions in their complex arguments. This in turn allows to extend the domain of analyticity once more, and one obtains analytic functions defined in the Bargmann-Hall-Wightman domain. This huge domain contains the "Euclidean points" $z_i = (i\tau_i, \vec{x}_i)$ with real τ_i, $\vec{x}_i$. Considered as functions of $\xi_i := (\vec{x}_i, \tau_i)$, the Wick rotated functions are the "Schwinger functions" $S_n(\xi_1, \ldots \xi_n)$, which are symmetric, analytic at $\xi_i \neq \xi_j$, and invariant under the Euclidean group.

It is convenient to "collect" all Schwinger functions in a generating functional

$$S[j] := \sum \frac{1}{n!} \int \Big(\prod d\xi_i \; j(\xi_i)\Big) \, S_n(\xi_1, \dots \xi_n) \equiv \Big\langle \mathrm{e}^{\int d\xi \; \phi_E(\xi) j(\xi)} \Big\rangle. \tag{1.2}$$

Knowledge of $S[j]$ is equivalent to the knowledge of the Schwinger functions, because the latter are obtained by variational derivatives,

$$S_n(\xi_1, \dots \xi_n) = \prod_i \frac{\delta}{\delta j(\xi_i)} S[j]|_{j=0}. \tag{1.3}$$

The generating functional for the "truncated (connected) Schwinger functions" $S_n^T(\xi_1, \dots \xi_n)$ (products of lower correlations subtracted) is $S^T[j] = \log S[j]$.

It should be emphasized that Fourier transformation, Lorentz invariance, and energy positivity enter the Wick rotation in a crucial way, so that in general curved spacetime, where none of these features is warranted, anything like the Wick rotation may by no means be expected to exist. Hence, we have

Lesson 1. *Euclidean QFT is a meaningful framework, related to some real-time QFT, only provided there is sufficient spacetime symmetry to establish the existence of a Wick rotation.*

AdS is a spacetime where the Wick rotation *can* be established [4]. The reason is that AdS may be viewed as a warped product of Minkowski spacetime $\mathbb{R}^{1,D-1}$ with $\mathbb{R}_+$, and the AdS group contains the Poincaré group. Namely, AdS is the hyperbolic surface in $\mathbb{R}^{2,D}$ given by $X \cdot X = 1$ in the metric of $\mathbb{R}^{2,D}$. In Poincaré coordinates,

$$X = \left(\frac{z}{2} + \frac{1 - x_\mu x^\mu}{2z}, \frac{x^\mu}{z}, -\frac{z}{2} + \frac{1 + x_\mu x^\mu}{2z}\right) \qquad (z > 0). \tag{1.4}$$

In these coordinates, the metric is

$$ds^2 = z^{-2}(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \tag{1.5}$$

hence for each fixed value of z, it is a multiple of the Minkowski metric.

The group $SO(2,D)$ of isometries of AdS is also the conformal group of Minkowski spacetime $\mathbb{R}^{1,D-1}$. The z-preserving subgroup acts on the coordinates x^μ like the Poincaré group. (The rest of the group are transformations which act non-linearly on the coordinates z and x in such a way that the boundary $z=0$ is preserved, and its points $(z=0,x)$ transform like scale and special conformal transformations of x.)

Thus, the Wick rotation can be performed in the variables x^μ alone, leading to the "Euclidean points" = points of Euclidean AdS

$$\Xi = \left(-\frac{z}{2} + \frac{1-|\xi|^2}{2z}, \frac{\xi^\mu}{z}, \frac{z}{2} + \frac{1+|\xi|^2}{2z} \right) \qquad (z>0), \tag{1.6}$$

which satisfy $\Xi \cdot \Xi = 1$ in the metric of $\mathbb{R}^{1,D+1}$.

1.2 Reconstruction and positivity

By famous reconstruction theorems [19, 17], the Wightman distributions or the Schwinger functions completely determine the quantum field, including its Hilbert space. For the reconstruction of the Hilbert space, one *defines* the scalar product between improper states $\phi(x_1)\dots\phi(x_n)\Omega$ to be given by the Wightman distributions. Therefore, the following positivity property of these distributions is absolutely crucial: Let $P=P[\phi_q]$ denote any polynomial in smeared fields. Then one has the positivity

$$(\Omega, P^*P\Omega) = ||P\Omega||^2 \geq 0. \tag{1.7}$$

(It could be zero because, e.g., P contains a commutator at spacelike distance such that $P=0$, or the Fourier transforms of the smearing functions avoid the spectrum of the four momenta such that $P\Omega = 0$.) On the other hand, inserting the smeared fields for P, $(\Omega, P^*P\Omega)$ is a linear combination of smeared Wightman distributions. Thus, every linear combination of smeared Wightman distributions which can possibly arise in this way must be non-negative.

This property translates, via the Wick rotation, into a property called "reflection positivity" of the Schwinger functions: Let $P = P[\phi_E]$ denote a polynomial in Euclidean fields smeared in a halfspace $\tau_i > 0$, and $\theta(P)$ the same polynomial smeared with the same functions reflected by $\tau_i \mapsto -\tau_i$. Then

$$\left\langle \theta(P)^* P \right\rangle \geq 0. \tag{1.8}$$

This expression is a linear combination of smeared Schwinger functions. Reflection positivity means that every linear combination which can possibly arise in this way must be non-negative.

As an example for the restrictivity of reflection positivity, we consider the 2-point function of a Euclidean conformal scalar field of scaling dimension Δ, $S_2(\xi_1, \xi_2) = |\xi_1 - \xi_2|^{-2\Delta}$. Ignoring smearing, we choose $P[\phi_E] = \phi_E(\frac{\tau}{2}, 0) - \phi_E(\frac{\tau}{2}, x)$ and obtain

$$\left\langle \theta(P)^* P \right\rangle = 2 \left[\tau^{-2\Delta} - (\tau^2 + x^2)^{-\Delta} \right]. \tag{1.9}$$

Obviously, this is positive iff $\Delta > 0$. This is the unitarity bound for conformal fields in 2 dimensions. (More complicated configurations of Euclidean points in $D > 2$ dimensions give rise to the stronger bound $\Delta \geq \frac{D-2}{2}$.)

The positivity requirements (1.7) resp. (1.8) are crucial for the reconstructions of the real-time quantum field, which start with the construction of the Hilbert space by defining scalar products on suitable function spaces in terms of Wightman or Schwinger functions of the form (1.7) resp. (1.8).

As conditions on the Wightman or Schwinger functions, the positivity requirements are highly nontrivial. It is rather easy to construct Wightman functions which satisfy all the requirements except positivity, and it is even more easy to guess funny Schwinger functions which satisfy all the requirements except reflection positivity. In fact, the remaining properties are only symmetry, Euclidean invariance, and some regularity and growth properties, which one can have almost "for free".

But without the positivity, these functions are rather worthless. From non-positive Wightman functions one would reconstruct fields without a probability interpretation, and reconstruction from non-positive Schwinger functions would not even yield locality and positive energy, due to the subtle way the properties intervene in the Wick rotation. In particular, the inverse Wick rotation uses methods from operator algebras which must not be relied on in "Hilbert spaces" with indefinite metric.

Lesson 2. *Schwinger functions without reflection positivity have hardly any physical meaning.*

1.3 Functional integrals

The most popular way to obtain Schwinger functions which are *at least in a formal way* reflection-positive, is via functional integrals [11]: the generating functional is

$$S[j] := Z^{-1} \int D\phi \, \mathrm{e}^{-I[\phi]} \cdot \mathrm{e}^{\int d\xi\, \phi(\xi) j(\xi)}, \tag{1.10}$$

where $I[\phi]$ is a Euclidean action of the form $\frac{1}{2}(\phi, A\phi) + \int d\xi V(\phi(\xi))$ with a quadratic form A which determines a free propagator, and an interaction potential $V(\phi)$. The normalization factor is $Z = \int D\phi \, \mathrm{e}^{-I[\phi]}$.

Consequently, the Schwinger functions are

$$S_n(\xi_1, \dots \xi_n) := Z^{-1} \int D\phi \; \phi(\xi_1) \dots \phi(\xi_n) \, \mathrm{e}^{-I[\phi]}. \tag{1.11}$$

Thus, one may think of them as the moments

$$S_n(\xi_1, \dots \xi_n) = \Big\langle \phi_E(\xi_1) \dots \phi_E(\xi_n) \Big\rangle, \tag{1.12}$$

of random variables $\phi_E(\xi)$, such that the functional integration variables ϕ are the possible values of ϕ_E with the probability measure $D\mu[\phi] = Z^{-1} D\phi \, \mathrm{e}^{-I[\phi]}$. (That Schwinger functions are moments of a measure, i.e., their representability by a functional integral, is not necessary by general principles, and this property usually gets lost when they are extended as distributions to coinciding points.)

The difficult part in constructing a Euclidean QFT along these lines is, of course, to turn the formal expressions (1.10) or (1.11) into well-defined quantities [12, 11]. This problem can be attacked in several different ways (e.g., perturbative or lattice approximations, or phase space cutoffs of the measure) which all involve the renormalization of formally diverging quantities. We shall by no means enter the problem(s) of renormalization in these lectures, but we emphasize

> **Lesson 3.** *The challenge of constructive QFT via functional integrals is to define the measure, in such a way that its formal benefits are preserved.*

Not the least among the "formal benefits" is reflection positivity which, as we have seen, is necessary to entail locality, energy positivity, and Hilbert space positivity for the reconstructed real-time field. Let us display the formal argument why the prescription (1.11) fulfills reflection positivity. It consists in separating the interaction part from the quadratic part, and splitting

$$\mathrm{e}^{-\int d\xi\, V(\phi(\xi))} = \mathrm{e}^{-\int_{\tau<0} d\xi\, V(\phi(\xi))} \cdot \mathrm{e}^{-\int_{\tau>0} d\xi\, V(\phi(\xi))} \equiv \theta(F)^* F \tag{1.13}$$

with $\xi = (\vec{x}, \tau)$ and $F = F[\phi] = \mathrm{e}^{-\int_{\tau>0} d\xi\, V(\phi(\xi))}$. Then

$$\Big\langle \theta(P)^* P \Big\rangle = \Big\langle \theta(FP)^* FP \Big\rangle_0 \tag{1.14}$$

where $\langle \ldots \rangle_0$ is the Gaussian expectation value defined with the quadratic part $\frac{1}{2}(\phi, A\phi)$ of the action, which is assumed to fulfill reflection positivity. Viewing F as an exponential series of smeared field products, $\langle \theta(FP)^* FP \rangle_0$ and hence $\langle \theta(P)^* P \rangle$ is positive. We see that it is important that the potential is "local" in the sense that it depends only on the field at a single point, in order to allow the split (1.13) into positive and negative Euclidean "time".

Even with the most optimistic attitude towards Lesson 3 ("nothing goes wrong upon renormalization"), we shall retain from Lesson 2 as a guiding principle:

Lesson 4. *A functional integral should not be trusted as a useful device for QFT if it violates reflection positivity already at the formal level.*

1.4 Semiclassical limit and large N limit

For later reference, we mention some facts concerning the effect of manipulations of generating functionals (irrespective how they are obtained) on reflection positivity of the Schwinger functions.

The product $S[j] = S^{(1)}[j]S^{(2)}[j]$ of two (or more) reflection-positive generating functional is another reflection-positive generating functional. In fact, because the truncated Schwinger functions are just added, the reconstructed quantum field equals $\phi^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes \phi^{(2)}$ defined on $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, or obvious generalizations thereof for more than two factors. In particular, positivity is preserved if $S[j]$ is raised to a power $\nu \in \mathbb{N}$.

The same is not true for a power $1/\nu$ with $\nu \in \mathbb{N}$: a crude way to see this is to note that reflection positivity typically includes as necessary conditions inequalities among truncated Schwinger n-point functions S^T_n of the general structure $S^T_4 \leq S^T_2 S^T_2$, while raising $S[j]$ to a power p amounts to replace S^T by $p \cdot S^T$.

This remark has a (trivial) consequence concerning the semiclassical limit: let us reintroduce the unit of action $\hbar$ and rewrite

$$S[j] = Z^{-1} \int D\phi \, \mathrm{e}^{-\frac{1}{\hbar} I[j;\phi]} \tag{1.15}$$

where $I[j;\phi] = I[\phi] - \int \phi j$ is the action in the presence of a source j. Appealing to the idea that when $\hbar$ is very small, the functional integral is sharply peaked around the classical minimum $\phi_{\text{s-cl}} = \phi_{\text{s-cl}}[j]$ of this action, let us replace $\hbar$ by $\hbar/\nu$ and consider the limit $\nu \to \infty$. Then we may expect (up to irrelevant constants)

$$S_{\text{s-cl}}[j] := \mathrm{e}^{-\frac{1}{\hbar} I[j;\phi_{\text{s-cl}}[j]]} = \lim_{\nu \to \infty} \left[\int D\phi \, \mathrm{e}^{-\frac{\nu}{\hbar} I[j;\phi]} \right]^{1/\nu}. \tag{1.16}$$

This generating functional treated perturbatively, gives the tree level (semiclassical) approximation to the original one, all loop diagrams being suppressed by additional powers of $\hbar/\nu$.

The functional integral in square brackets is "as usual" with $\hbar/\nu$ in place of $\hbar$, hence we may assume that it satisfies reflection positivity. But we have no reason to expect $S_{\text{s-cl}}[j]$ to be reflection-positive, because of the presence of the power $1/\nu$. Thus $S_{\text{s-cl}}[j]$ does not generate reflection-positive Schwinger functions, and hence no acceptable quantum field. This is, clearly, no surprise, because a classical field theory is not a quantum field theory.

A variant of this argument is less trivial, concerning the large N limit. If one raises $S[j]$ to some power N, the truncated Schwinger functions are multiplied by the factor N, and diverge as $N \to \infty$. Rescaling the field by $N^{-\frac{1}{2}}$ stabilizes the 2-point function (assuming the 1-point function $\langle \phi_E \rangle$ to vanish), but suppresses all higher truncated n-point functions, so that the limit $N \to \infty$ becomes Gaussian, i.e., one ends up with a free field. To evade this conclusion, one has to "strengthen" the interaction at the same time to counteract the suppression of higher truncated correlations. Let us consider $S[j]$ of the functional integral form. Raising S to the power N, amounts to integrate over N independent copies of the field ($D^N\underline{\phi} = D\phi_1 \dots D\phi_N$) with interaction $V(\underline{\phi}) = \sum_i V(\phi_i)$ and coupling to the source $j \cdot \sum \phi_i$. One way to strengthen the interaction is to replace, e.g., $V(\phi) = \lambda \sum_i \phi_i^4$ by $V(\phi) = \lambda(\sum_i \phi_i^2)^2$ giving rise to much more interaction vertices coupling the N previously decoupled copies of the field among each other. At the same time, the action acquires an $O(N)$ symmetry, so one might wish to couple the sources also only to $O(N)$ invariant fields, and replace the source term by $j \cdot \sum \phi_i^2$, hence

$$I_N[j, \underline{\phi}] = \frac{1}{2}(\underline{\phi}, A\underline{\phi}) + \int \lambda(\underline{\phi}^2)^2 + \int j \cdot \underline{\phi}^2. \qquad (1.17)$$

We call the resulting functional integral $S_N[j]$.

All these manipulations maintain the formal reflection positivity of $S_N[j]$ at any finite value of N. An inspection of the Feynman rules for the perturbative treatment shows that now all truncated n-point functions still carry an explicit factor of N, and otherwise have a power series expansion in N and λ where each term has less powers of N than of λ. Introducing the 't Hooft coupling $\theta = N\lambda$, this yields an expansion in θ and $1/N$. Fixing θ and letting $N \to \infty$, suppresses the $1/N$ terms, so that the asymptotic behaviour at large N is

$$S_N[j] \sim \mathrm{e}^{N[S^T_\infty(\theta)+O(1/N)]}. \tag{1.18}$$

To obtain a finite non-Gaussian limit, one has to take

$$S_\infty[j] := \lim_{N\to\infty} S_N[j]^{1/N} = \mathrm{e}^{S^T_\infty(\theta)}. \tag{1.19}$$

But this reintroduces the fatal power $1/N$ which destroys reflection positivity. According to Lesson 4, this means

Lesson 5. *The large N limit of a QFT is* not *itself a QFT.*

It is rather some classical field theory, for the same reason as before: namely the explicit factor N combines with the tacit inverse unit of action $1/\hbar$ in the exponent of (1.18) to the inverse of an "effective" unit of action $\hbar/N \to 0$. What large N QFT has to say about QFT, is the (divergent) asymptotic behaviour of correlations as N gets large.

2 Lecture 2: AdS-CFT

2.1 A positivity puzzle

The AdS-CFT correspondence, which provides the generating functional for conformally invariant Schwinger functions from a classical action I on AdS, was given by Witten [20] and Polyakov *et al.* [14] as a "model" for Maldacena's Conjecture. We shall discuss this formula in the light of the previous discussions about QFT, in which it appears indeed rather puzzling.

The formula is essentially classical, because it is supposed to capture only the infinite N limit of the Maldacena conjecture.

The general structure of the formula is

$$S^{\text{AdS-CFT}}_{\text{s-cl}}[j] := \mathrm{e}^{-I[\phi^{\text{AdS}}[j]]} \tag{2.1}$$

where $I[\phi^{\text{AdS}}]$ is an AdS-invariant action of a field on AdS, and $\phi^{\text{AdS}}[j]$ is the (classical) minimum of the action I under the restriction that ϕ^{AdS} has prescribed boundary values j. More precisely, introducing the convenient Poincaré coordinates $(z > 0, \xi \in \mathbb{R}^D)$ of Euclidean AdS such that the boundary $z = 0$ is identified with D-dimensional Euclidean space, it is required that the limit

$$\partial\phi^{\text{AdS}}(\xi) := \lim_{z\to 0} z^{-\Delta}\phi^{\text{AdS}}(z,\xi) \tag{2.2}$$

exists, and coincides with a prescribed function $j(\xi)$.

It follows from the AdS-invariance of the action $I[\phi^{\text{AdS}}]$ (and the assumed AdS-invariance of the functional measure) that the variational derivatives of $S^{\text{AdS-CFT}}_{\text{s-cl}}[j]$ with respect to the source j are conformally covariant functions, more precisely, they transform like the correlation functions of a Euclidean conformal field of scaling dimension ("weight") Δ. Thus, symmetry and covariance are automatic. But how about reflection positivity?

To shed light on this aspect [7], we appeal again to the idea that a functional integral is sharply peaked around the minimum of the action, when the unit of action becomes small, and rewrite $S[j]$ as

$$S^{\text{AdS-CFT}}_{\text{s-cl}}[j] = \lim_{\nu\to\infty}\Big[\int D\phi^{\text{AdS}}\mathrm{e}^{-\nu I[\phi^{\text{AdS}}]}\cdot\delta\left[\partial\phi^{\text{AdS}} - j\right]\Big]^{1/\nu} \tag{2.3}$$

where a formal functional δ-function restricts the integration to those field configurations whose boundary limit (2.2) exists and coincides with the given function $j(\xi)$. We see that ν takes the role of the inverse unit of action $1/\hbar$ in (2.3), so that $\nu\to\infty$ signals the classical nature of this limit, hence of the original formula.

Now, there are two obvious puzzles concerning formal reflection positivity of this generating functional. The first is the same which was discussed in Sect. 1.4, namely the presence of the inverse power $1/\nu$, which arises due to the classical nature of the formula. Even if the functional integral in square brackets were positive, this power most likely would spoil this property. (In fact, the correlation functions obtained from $S^{\text{AdS-CFT}}_{\text{s-cl}}$ can be seen explicitly to have logarithmic rather than power-like short-distance singularities, and hence manifestly violate positivity [15].)

The obvious cure (as it is of course also suggested in the original papers [20, 14]) is to interpret the AdS-CFT formula (2.1) only as a semiclassical approximation to the "true" (quantum) formula, and consider instead the quantum version

$$\left\langle \mathrm{e}^{\int d\xi\ \phi_E^{\text{AdS-CFT}}(\xi) j(\xi)} \right\rangle \equiv S^{\text{AdS-CFT}}[j] := \int D\phi^{\text{AdS}}\ \mathrm{e}^{-I[\phi^{\text{AdS}}]} \cdot \delta\left[\partial\phi^{\text{AdS}} - j\right] \tag{2.4}$$

as the generating functional of conformally invariant Schwinger functions of a Euclidean QFT on $\mathbb{R}^D$.

But the second puzzle remains: for this expression, the formal argument for reflection positivity of functional integrals, presented in Sect. 1.3, fails: that argument treats the exponential of the interaction part of the action as a field insertion in the functional integrand, and it was crucial that field insertions ϕ in the functional integral amount to the same insertions of the random variable ϕ_E in the expectation value $\langle \dots \rangle$, achieved by variational derivatives of the generating functional S with respect to the source j. But this property (1.11) is not true for the AdS-CFT functional integral $S^{\text{AdS-CFT}}$ where the coupling to the source is via a δ-functional rather than an exponential!

So why should one believe that the quantum AdS-CFT generating functional satisfies reflection positivity, so as to be acceptable for a conformal QFT on the boundary? Surprisingly enough, explicit studies of AdS-CFT Schwinger functions, computing the operator product expansion coefficients of the 4-point function at tree level [15], show no signs of manifest positivity violation which could not be restored in the full quantum theory (i.e., regarding the logarithmic behaviour as first order terms of the expansion of anomalous dimensions). Why is this so?

An answer is given [7] by a closer inspection of the Feynman rules which go with the functional δ function in the perturbative treatment of the functional integral. For simplicity, we consider a single scalar field with quadratic Klein-Gordon action $\int \phi^{\rm AdS}(-\Box + M^2)\phi^{\rm AdS}$ and a polynomial self-interaction. As usual, the Feynman diagrams for truncated n-point Schwinger functions are connected diagrams with n exterior lines attached to the boundary points ξ_i, and with vertices according to the polynomial interaction and internal lines connecting the vertices. Each vertex involves an integration over AdS. (For our considerations it is more convenient to work in configuration space rather than in momentum space.) However, the implementation of the functional δ-function, e.g., by the help of an auxiliary field: $\delta(\partial\phi^{\rm AdS} - j) = \int Db\, {\rm e}^{i\int b(\xi)(\partial\phi^{\rm AdS}(\xi) - j(\xi))}$, modifies the propagators. One has the bulk-to-bulk propagator $\Gamma(z, \xi; z', \xi')$ connecting two vertices, the bulk-to-boundary propagator $K(z, \xi; \xi')$ connecting a boundary point with a vertex, and the boundary-to-boundary propagator $\beta(\xi; \xi')$ which coincides with the tree level 2-point function.

The precise determination of these propagators gives the following result.

Γ equals the Green function G_+ of the Klein-Gordon operator which behaves $\sim z^{\Delta_+}$ near the boundary, where

$$\Delta_\pm = \frac{D}{2} \pm \sqrt{\frac{D^2}{4} + M^2}. \tag{2.5}$$

It is a hypergeometric function of the Euclidean AdS distance. K is a multiple of the boundary limit $\lim_{z'\to 0} z'^{-\Delta_\cdot}$ in the variable z' of $G_+(z, \xi; z', \xi')$, and β is a multiple of the double boundary limit in both variables z and z' of G_+ [1]:

$$\Gamma = G_+, \qquad K = c_1 \cdot \lim_{z\to 0} z^{-\Delta_+} G_+, \qquad \beta = c_2 \cdot \lim_{z\to 0} z^{-\Delta_+} \lim_{z'\to 0} z'^{-\Delta_+} G_+ \tag{2.6}$$

with certain numerical constants c_1 and c_2. Specifically [7],

$$c_1 = 2\Delta_+ - D = \sqrt{D^2 + 4M^2}, \tag{2.7}$$

and, as will be crucial for the sequel,

$$c_2 = c_1^2. \tag{2.8}$$

Now, let us consider the conventional (as in Sect. 1.3) functional integral for a Euclidean field on AdS

$$S^{\rm AdS}[J] = Z^{-1} \int D\phi^{\rm AdS} {\rm e}^{-I[\phi^{\rm AdS}]} {\rm e}^{\int \sqrt{g}\, \phi^{\rm AdS} J^{\rm AdS}}, \tag{2.9}$$

choosing $G_+(z,\xi;z',\xi')$ as the propagator defining the Gaussian functional measure. Its perturbative Schwinger functions are sums over ordinary Feynman graphs with all lines given by G_+. Taking the simultaneous boundary limits $\lim_{z_i\to 0} z_i^{-\Delta_+}(\cdot)$ of the Schwinger functions in all their arguments, one just has to apply the boundary limit to the external argument of each external line. This yields bulk-to-bulk, bulk-to-boundary and boundary-to-boundary propagators

$$G_+, \qquad H_+ = \lim_{z\to 0} z^{-\Delta_+} G_+, \qquad \alpha_+ = \lim_{z\to 0} z^{-\Delta_+} \lim_{z\to 0} z'^{-\Delta_+} G_+. \tag{2.10}$$

Comparison of (2.6) and (2.10) implies for the resulting Schwinger functions

$$S_n^{\text{AdS-CFT}}(\xi_1,\dots,\xi_n) = c_1^n \cdot \left(\prod_i \lim_{z_i\to 0} z_i^{-\Delta_+} \right) S_n^{\rm AdS}(z_1,\xi_1,\dots z_n,\xi_n) \tag{2.11}$$

where it is crucial that $c_2 = c_1^2$ because each external end of a line must come with the same factor.

In other words, we have shown that the Schwinger functions generated by the functional integral (2.4) formally agree (graph by graph in unrenormalized perturbation theory) with the boundary limits of those generated by (2.9). The latter satisfy reflection positivity by the formal argument of Sect. 1.3, generalized to AdS. Taking the joint boundary limit preserves positivity, because this step essentially means that the smeared fields involved in P in (1.8) are smeared over the boundary $z = 0$ only. Thus, (2.4) indeed satisfies reflection positivity, inspite of its appearance.

Because the Wick rotation affecting the Minkowski coordinates commutes with the boundary limit in z, we conclude that the same relation (2.11) also holds for the Wightman functions, and hence for the reconstructed real time quantum fields:

$$\phi_q^{\text{AdS-CFT}}(x) = c_1 \cdot \partial\phi_q^{\text{AdS}}(x) \equiv c_1 \cdot \lim_{z^n \to 0} z^{-\Delta_+} \phi_q^{\text{AdS}}(z,x) \qquad (2.12)$$

$x \in D$-dimensional Minkowski spacetime. This relation describes the restriction of an AdS covariant field to its timelike boundary [3], and generalizes the well-known fact that Poincaré covariant quantum fields can be restricted to timelike hypersurfaces, giving rise to quantum fields in lower dimensions, see Sect. 3.1. Moreover, because the AdS field (formally) satisfies reflection positivity, so does its boundary restriction.

We have established the identification (2.11), (2.12) for symmetric tensor fields of arbitrary rank [13] (with arbitrary polynomial couplings), see the Appendix. Although we have not considered antisymmetric tensors nor spinor fields [6], there is reason to believe that this remarkable conclusion is true in complete generality.

Lesson 6. *Quantum fields defined by AdS-CFT are the boundary restrictions (limits) of AdS fields quantized conventionally on the bulk (with the same classical action).*

We want to mention that in the semiclassical approximation (2.1), one has the freedom to partially integrate the classical quadratic action and discard boundary contributions, which are of course quadratic in j and hence contribute only to the tree level 2-point function. This kind of ambiguity has been settled previously [10] by imposing Ward identities on the resulting correlation functions. The normalization c_2 of the tree level 2-point function, obtained quite naturally by the method mentioned above, precisely matches the normalization obtained by the Ward identity method.

Let us look at this from a different angle. Changing the tree level

2-point function amounts to multiplication of the generating functional by a Gaussian. Thus, any different normalization would add (as in Sect. 1.4) a Gaussian (free) field to the conformal Minkowski field $\partial\phi_q^{\mathrm{AdS}}$. Not surprisingly, the sum would violate Ward identities, which are satisfied by the field without the extra Gaussian.

3 Lecture 3: Brane restrictions and AdS-CFT

We want to discuss the results obtained by formal reasoning in the previous lecture, in the light of exact results on QFT.

3.1 Brane restrictions

Quantum fields may be restricted to timelike hypersurfaces [5]. This is a non-trivial statement since they are distributions which become operators only after smearing with smooth test functions, so it is not obvious that one may fix one of the spacetime coordinates to some value. Indeed, $t = 0$ fields in general do not exist due to renormalization. However, it is possible to fix one of the spacelike coordinates thanks to the energy positivity, by doing so in the analytically continued Wightman functions in the forward tube, which gives other analytic functions whose real-time limits $\mathrm{Im}\,(z_i - z_{i+1}) \searrow 0$ exist as distributions in a spacetime of one dimension less.

The restricted field inherits locality (in the induced causal structure of the hypersurface), Hilbert space positivity (because the Hilbert space does not change in the process), and covariance. However, only the subgroup which preserves the hypersurface may be expected to act geometrically on the restricted field.

This result, originally derived for Minkowski spacetime [5], has been generalized to AdS in [2]. Here, the warped product structure implies that each restriction to a $z = \text{const.}$ hypersurface ("brane") gives a Poincaré covariant quantum field in Minkowski spacetime. One thus obtains a family of such fields, $\phi_z(x) := \phi^{\mathrm{AdS}}(z, x)$, defined on the same Hilbert space. Moreover, because spacelike separation in the Minkowski coordinates alone implies spacelike separation in AdS, the fields of this family are mutually local among each other. Even more, $\phi_z(x)$ commute with $\phi_{z'}(x')$ also at timelike distance provided $(x-x')_\mu(x-x')^\mu < (z-z')^2$.

3.2 AdS → CFT as QFT on the limiting brane

Now assume in addition that the Wightman distributions $W_n^{\rm AdS}$ of a (scalar) quantum field on AdS admit a finite limit

$$\prod(\lim_{z_i\to 0} z_i^{-\Delta}) W_n^{\rm AdS}(z_1,x_1;\dots;z_n,x_n) =: W_n(x_1,\dots,x_n) \qquad (3.1)$$

for some value of Δ. It was proven [3] that these limits define a (scalar) Wightman field on Minkowski spacetime, which may be written as

$$\phi(x) = \lim_{z\to 0} z^{-\Delta}\phi^{\rm AdS}(z,x). \qquad (3.2)$$

In addition to the usual structures, this field inherits conformal covariance from the AdS covariance of $\phi^{\rm AdS}$, whose weight Δ emerges through the limit $\lim_{z\to 0} z^{-\Delta}(\cdot)$.

None of the fields ϕ_z ($z=$ const. $\neq 0$) is conformally covariant because its family parameter z sets a scale; hence the boundary limit may be re-interpreted as a scaling limit within a family of non-scale-invariant quantum fields.

Comparing the rigorous formula (3.2) with the conclusion (2.12) obtained by formal reasoning with unrenormalized Schwinger functions, we conclude

Lesson 7. *The prescription for the AdS-CFT correspondence coincides with a special instance of the general scheme of brane restrictions, admitted in QFT.*

3.3 AdS ← CFT by holographic reconstruction

In view of the preceding discussion, the inverse direction AdS ← CFT amounts to the reconstruction of an entire family of Wightman fields ϕ_z ($z \in \mathbb{R}_+$) from a single member $\phi_{z=0}$ of that family, with the additional requirement that two members of the family commute at spacelike distance in AdS which involves the family parameters z, z'. This is certainly a formidable challenge, and will not always be possible. We first want to illustrate this by a free field, and then turn to a more abstract treatment of the problem in the general case.

Let us consider [3, 8] a canonical Klein-Gordon field of mass M on AdS. The "plane wave" solutions of the Klein-Gordon equation are the functions

$$z^{D/2} J_\nu(z\sqrt{k^2})\mathrm{e}^{\pm ik\cdot x}, \tag{3.3}$$

where $\nu = \Delta - D/2 = \sqrt{D^2/4 + M^2}$, and the Minkowski momenta range over the entire forward lightcone V_+. It follows that the 2-point function is

$$\begin{aligned}\langle\Omega\phi^{\mathrm{AdS}}(z,x)\phi^{\mathrm{AdS}}(z',x')\Omega\rangle &\sim \\ &\sim (zz')^{D/2}\int_{V_+} d^Dk J_\nu(z\sqrt{k^2})J_\nu(z\sqrt{k^2})\mathrm{e}^{-ik(x-x')} \sim \\ &\sim (zz')^{D/2}\int_{\mathbb{R}_+} dm^2 J_\nu(zm)J_\nu(z'm)W_m(x-x')\end{aligned} \tag{3.4}$$

(ignoring irrelevant constants throughout), where W_m is the massive 2-point function in D-dimensional Minkowski spacetime.

Restricting to any fixed value of z, we obtain the family of fields $\phi_z(x)$ which are all different "superpositions" of massive Minkowski fields with Källen-Lehmann weights $d\mu_z(m^2) = dm^2 J_\nu(zm)^2$. Such fields are known as "generalized free fields". Using the asymptotic behavior of the Bessel functions $J_\nu(u) \sim u^\nu$ at small u, the boundary field ϕ_0 turns out to have the Källen-Lehmann weight $d\mu_0(m^2) \sim m^{2\nu}dm^2$.

In order to reconstruct $\phi_z(x)$ from $\phi_0(x)$, one has to "modulate" its weight function, which can be achieved with the help of a pseudo-differential operator:

$$\phi_z(x) \sim z^\Delta \cdot j_\nu(-z^2\Box)\phi_0(x) \tag{3.5}$$

where j_ν is the function $j_\nu(u^2) = u^{-\nu} J_\nu(u)$ on $\mathbb{R}_+$. Note that the operators $j_\nu(-z^2\Box)$ are highly non-local because $j_\nu(u)$ is not a polynomial, but they produce a family of fields which all satisfy local commutativity with each other at spacelike Minkowski distance [8].

In order to reconstruct a local field $\phi^{\text{AdS}}(z, x)$ on AdS, which fulfils local commutativity with respect to the causal structure of AdS, Minkowski locality is, however, not sufficient. A rather nontrivial integral identity for Bessel functions guarantees that $\phi_z(x)$ and $\phi_{z'}(x')$ commute even at timelike distance provided $(x - x')_\mu (x - x')^\mu < (z - z')^2$. Only this ensures that $\phi^{\text{AdS}}(z, x) := \phi_z(x)$ is a local AdS field.

We have seen that the reconstruction of a local AdS field from its boundary field is a rather nontrivial issue even in the case of a free field, and exploits properties of free fields which are not known how to generalize to interacting fields.

In the general case, there is an alternative algebraic reconstruction [18] of local AdS observables, which is however rather abstract and might not yield any fields in the Wightman sense. This approach makes use of the global action of the conformal group on the Dirac completion of Minkowski spacetime, and of a corresponding global coordinatization of AdS (i.e., unlike most of our previous considerations, it does not work in a single Poincaré chart (z, x)).

The global coordinates of AdS are

$$X = \left(\frac{1}{\cos\rho}\vec{e}, \frac{\sin\rho}{\cos\rho}\vec{E}\right) \tag{3.6}$$

where $\rho < \frac{\pi}{2}$ and $\vec{e}$ and $\vec{E}$ are a 2-dimensional and a D-dimensional unit vector, respectively. A parametrization of the universal covering of AdS is obtained by writing $\vec{e} = (\cos\tau, \sin\tau)$ and considering the timelike coordinate $\tau \in \mathbb{R}$. Thus, AdS appears as a cylinder $\mathbb{R} \times B^D$. While the metric diverges with an overall factor $\cos^{-2}\rho$ with $\rho \nearrow \frac{\pi}{2}$ as the boundary is reached, lightlike curves hit the boundary at a finite angle.

The boundary manifold has the structure of $\mathbb{R} \times S^{D-1}$, which is the universal covering of the conformal Dirac completion of Minkowski spacetime.

We consider causally complete regions $K \subset \mathbb{R} \times S^{D-1}$, and associate with them causally complete "wedge" regions $W(K) \subset \mathbb{R} \times B^D$, which are the causal completion of K in the causal structure of the bulk. It then follows that $W(K_1)$ and $W(K_2)$ are causal complements in the bulk of each other, or AdS transforms of each other, iff K_1 and K_2 are causal complements in the boundary of each other, or conformal transforms of each other, respectively.

Now, we assume that a CFT on $\mathbb{R} \times S^{D-1}$ is given. We want to define an associated quantum field theory on AdS. Let $A(K)$ be the algebras generated by CFT fields smeared in K. Then, by the preceding remarks, the operators in $A(K)$ have the exact properties as to be expected from AdS quantum observables localized in $W(K)$, namely AdS local commutativity and covariance. AdS observables in compact regions O of AdS are localized in every wedge which contains O, hence it is consistent to define [18]

$$A^{\mathrm{AdS}}(O) := \bigcap_{W(K) \supset O} A(K) \tag{3.7}$$

as the algebra of AdS observables localized in the region O. Because any two compact regions at spacelike AdS distance belong to some complementary pair of wedges, this definition in particular guarantees local commutativity. Note that this statement were not true, if only wedges within a Poincaré chart (z, x) were considered.

Lesson 8. *Holographic reconstruction is possible in general, but requires a global treatment in order to resolve possible causality paradoxes with AdS-CFT.*

The only problem with this definition is that the intersection of algebras might be trivial (in which case the AdS QFT has only wedge-localized observables). But when we know that the conformal QFT on the boundary arises as the restriction of a bulk theory, then the intersection of algebras (3.7) contains the original bulk field smeared in the region O.

3.4 Conformal perturbation theory via AdS-CFT

As we have seen, a Klein-Gordon field on AdS gives rise to a generalized free conformal field. Perturbing the former by an interaction, will perturb the latter. But perturbation theory of a generalized free field is difficult to renormalize, because there is a continuum of admissible counter terms associated with the continuous Källen-Lehmann mass distribution of the generalized free field.

This suggests to perform the renormalization on the bulk, and then take the boundary limit of the renormalized AdS field. Preserving AdS symmetry, drastically reduces the free renormalization parameters.

This program is presently studied [9]. Two observations are emerging: first, to assume the existence of the boundary limit of the remormalized AdS field constitutes a nontrivial additional renormalization condition; and second, the resulting renormalization scheme for the boundary field differs from the one one would have adopted from a purely boundary (Poincaré invariant) point of view.

We do not enter into this in more detail [9]. Let us just point out that this program can be successful only for very special interactions of the conformal field, which "derive" from local AdS interactions.

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