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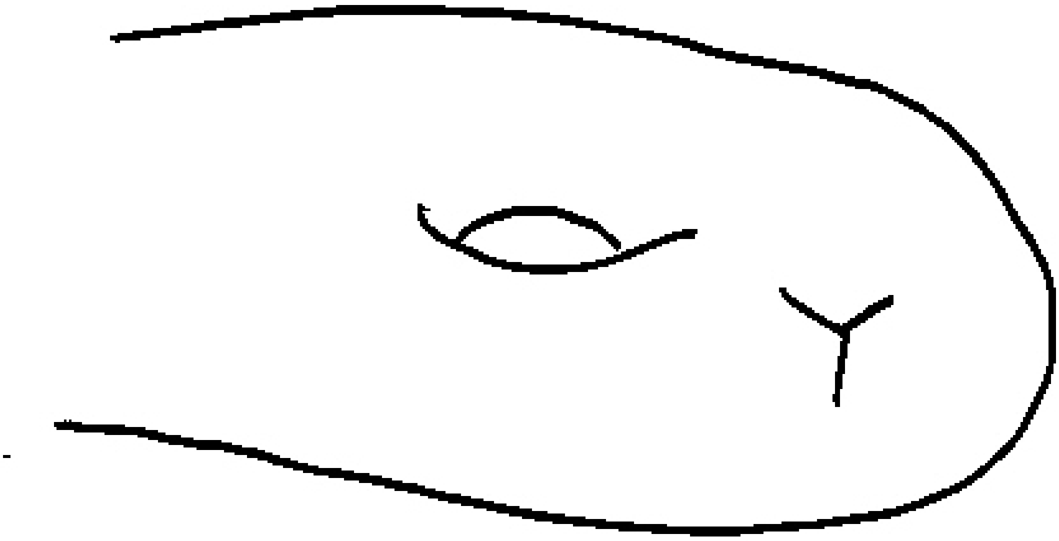
"FLOER HOMOLOGY IN CLASSICAL MECHANICS AND QUANTUM FIELD THEORY

TOPOLOGICAL QUANTUM FIELD THEORY IN DIMENSION d IS A FUNCTOR

$$Z : \left\{ \begin{array}{l} d\text{-DIM. COMPACT} \\ \text{ORIENTED} \\ \text{SMOOTH} \\ \text{MANIFOLDS} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{COMPLEX} \\ \text{VECTOR} \\ \text{SPACES} \end{array} \right\}$$

$$\left\{ \begin{array}{l} (d+1)\text{-DIM. ORIENTED} \\ \text{SMOOTH MANIFOLDS} \\ \text{(POSSIBLY) WITH} \\ \text{BOUNDARY} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{VEC-} \\ \text{TORS} \end{array} \right\}$$



$\dim \Sigma = d\,, \quad \dim Y = d+1$

$\partial Y = \Sigma$

$Z(\Sigma)$ – VECTOR SPACE

$Z(Y)$ – VECTOR IN $Z(\Sigma)$

AXIOMS:

(A1) $Z(\Sigma^*) = Z(\Sigma)^*$

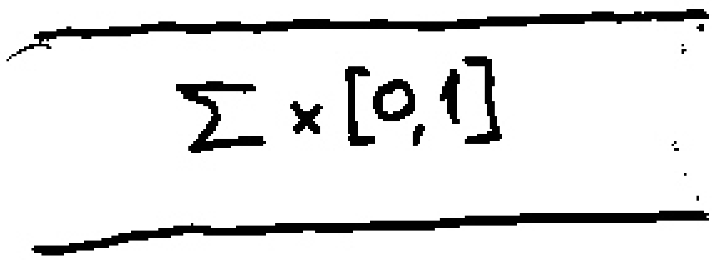
(A2) $Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$

$\cong \mathrm{Hom}(Z(\Sigma_1)^*, Z(\Sigma_2))$

(A3) $Z(Y_1 \cup_{\Sigma_2} Y_2) = Z(Y_2) \circ Z(Y_1)$

(A4) $Z(\emptyset) = \mathbb{C}$

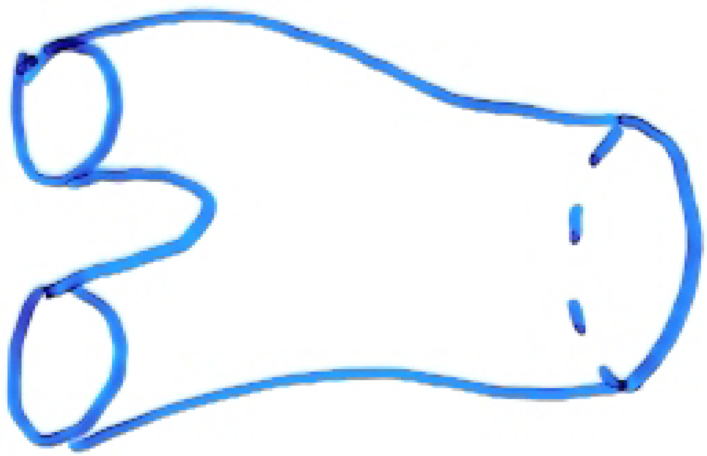
(A5) $Z(\Sigma \times [0,1]) = \mathrm{Id} : Z(\Sigma) \to Z(\Sigma)$



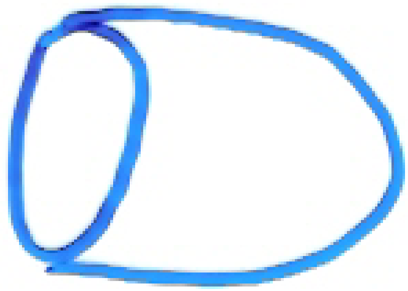
CASE $d=1$

(1.) $\Sigma \cong S^1$, $Z(S^1) =: H$

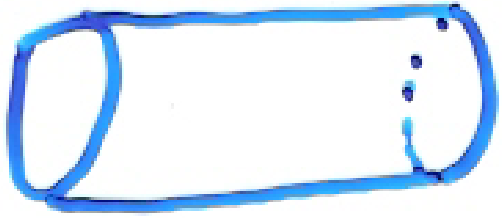
(2.) EVERY 2-DIM. Y CAN BE DECOMPOSED TO



PANTS

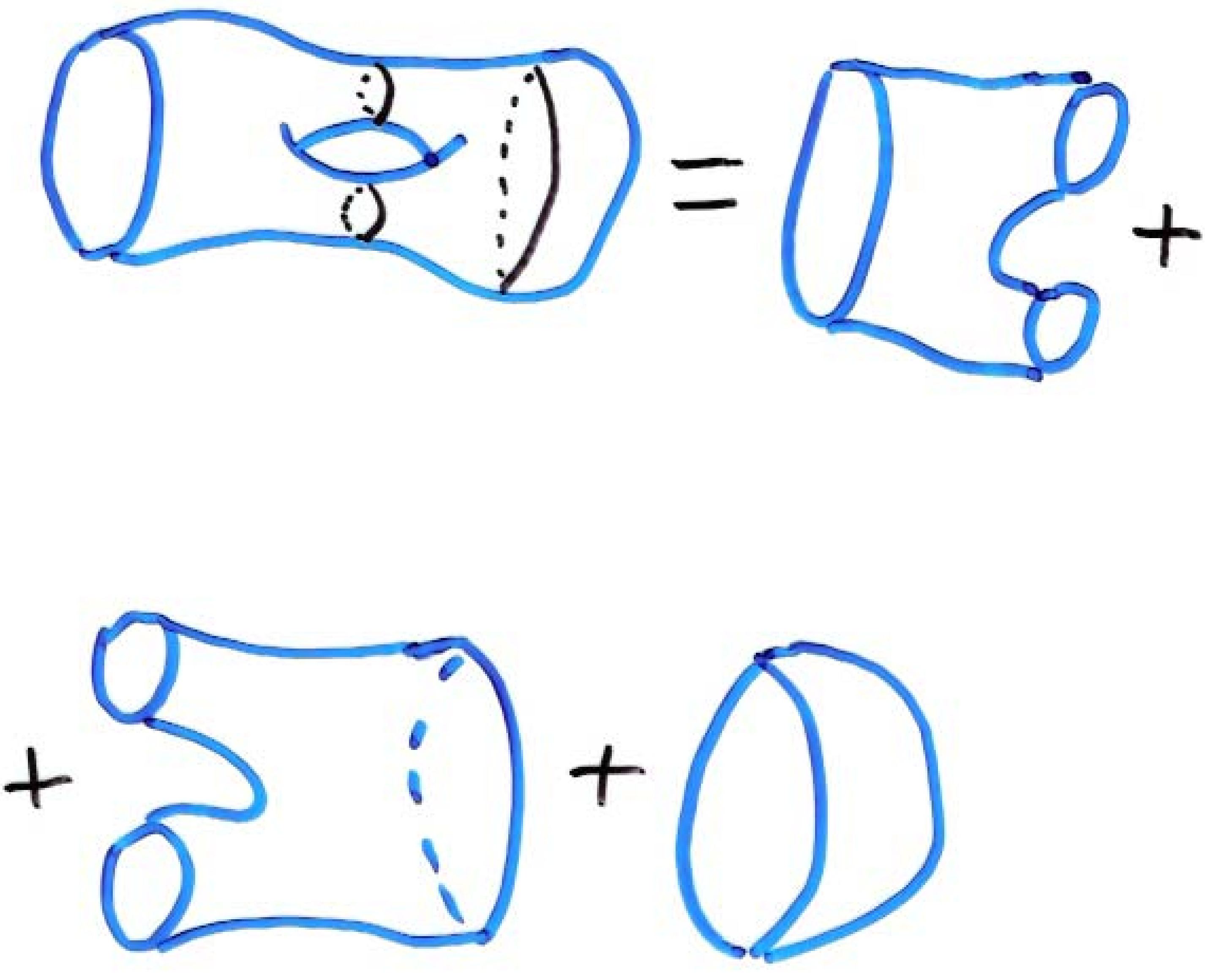


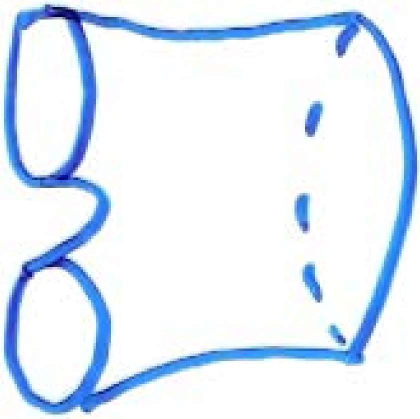
CAPS



CYLINDERS

EXAMPLE

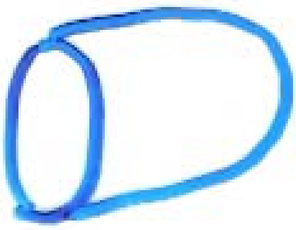




$$H \otimes H \to H$$

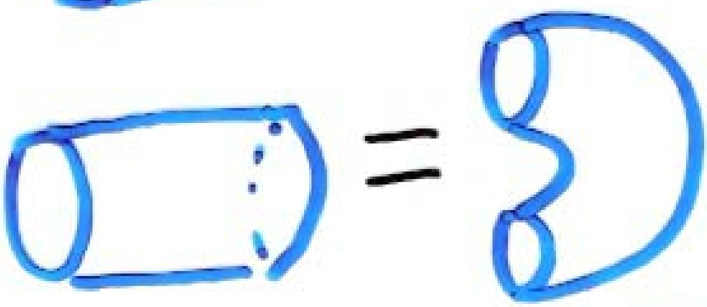
$$a \otimes b \mapsto a * b$$

(MULTIPLICATION)



$$\theta : H \to \mathbb{C}$$

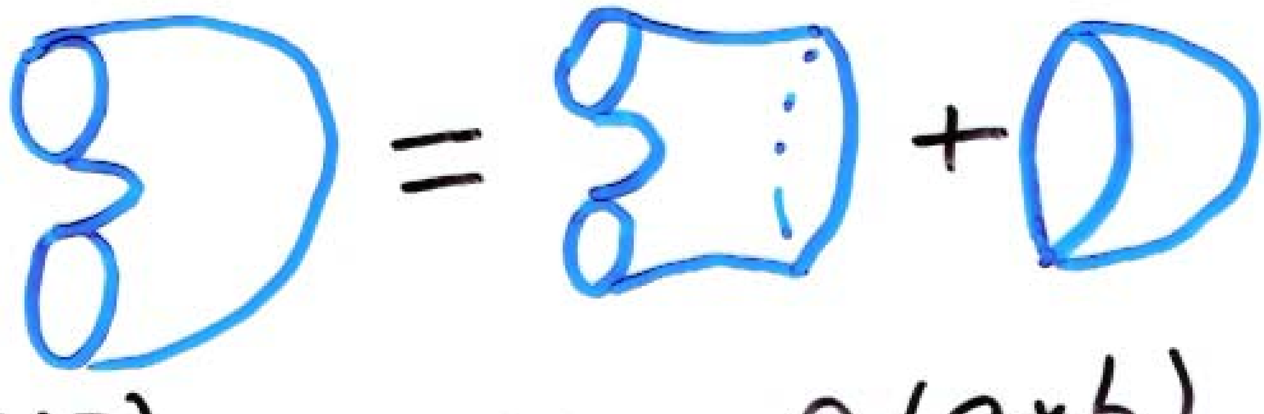
$(\theta \in H^*,\ $ 1-FORM$)$



$$H \otimes H \longrightarrow \mathbb{C}$$

$$a \otimes b \mapsto \langle a, b \rangle$$

(SCALAR PRODUCT)



$$\overset{(A3)}{\Longrightarrow} \langle a, b \rangle = \theta(a * b)$$

(FROBENIUS ALGEBRA)

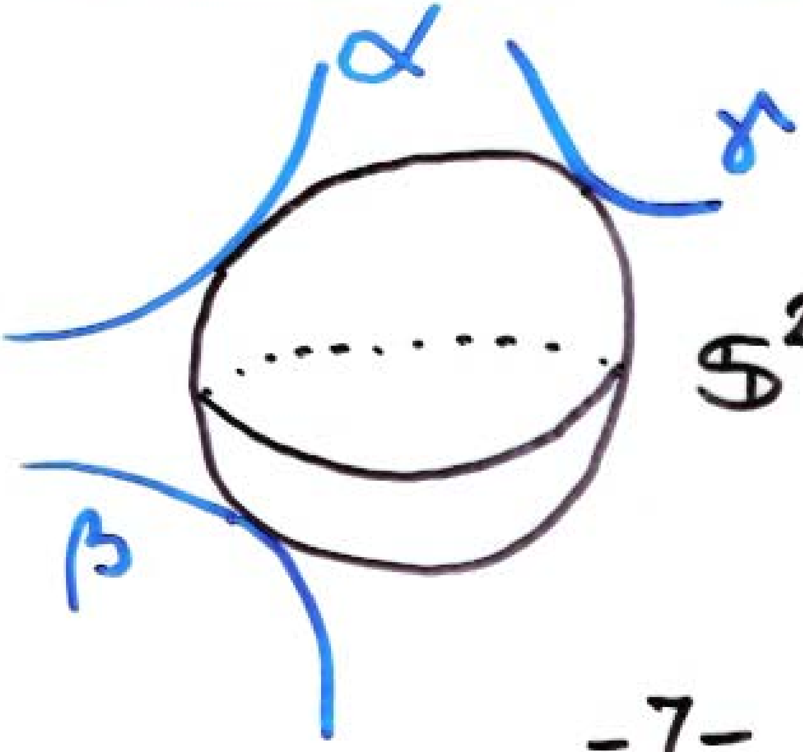
TWO EXAMPLES

M - COMPACT SYMPLECTIC MANIFOLD

① QUANTUM COHOMOLOGY

$\alpha, \beta, \gamma \in H_*(M)$, $A \in H_2(M)$

GROMOV-WITTEN INVARIANTS $\Phi_A(\alpha, \beta, \gamma)$



S^2 - HOLOMORPHIC SPHERE

$$QH^*(M) = H^*(M) \otimes \mathbb{Z}[q, q^{-1}]$$

QUANTUM PRODUCT:

$a, b, c \in QH^*(M)$

$$a = \sum_i a_i q^i$$

$$b = \sum_j b_j q^j$$

$$c = \sum_k c_k q^k$$

$$\langle a * b, c \rangle =$$

$$= \sum_{i,j,k} \sum_{A \in H_2(M)} \phi_A(PD(a_i), PD(b_j), PD(c_k))$$

② FLOER COHOMOLOGY

$H : M \times [0,1] \longrightarrow \mathbb{R}$ HAMILTONIAN

$HF^*(M, H)$ - COHOMOLOGY

GROUP GENERATED BY

TIME-ONE PERIODIC

ORBITS OF H.

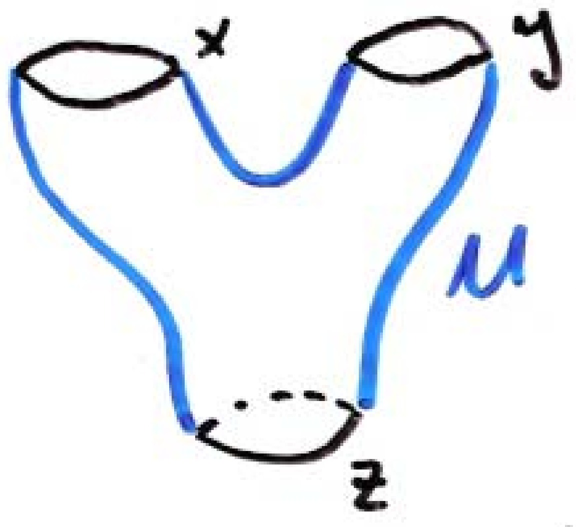
IT HAS A PRODUCT.

PERTURBED CAUCHY-RIEMANN EQUATIONS: $\boxed{\bar{\partial}_J u = -\nabla H(u)}$

("GRADIENT LINES" OF

$\mathcal{A}_H = \int p\,dq - H\,dt$)

CO-BOUNDARY OPERATOR (DIFFERENTIAL)



PRODUCT ON $HF^*(M,H)$

"THEOREM":

$$\boxed{HF^*(M,H) \cong QH^*(M)}$$

TWO APPLICATIONS TO CLASSICAL MECHANICS:

1.) FLOER'S PROOF OF ARNOLD CONJECTURE

$$\#\mathrm{Periodic}(H) \geq \sum_k \beta_k(M)$$

2.) WHAT IS THE MINIMAL AMOUNT OF ENERGY REQUIRED TO GENERATE A GIVEN HAMILTONIAN DIFFEOMORPHISM

$$\phi : M \longrightarrow M \quad ?$$

$$\ell(\phi_t^H) = \int_0^1 (\max H_t(x) - \min H_t(x))\, dt$$

$$E(\phi) = \inf_{H \mapsto \phi = \phi_1^H} \ell(\phi_t^H)$$

HOFER, LALONDE-McDUFF

$$\boxed{d(\phi, \psi) = E(\phi \circ \psi^{-1})}$$

IS A DISTANCE.

GENERALIZATION TO LAGRANGIAN SUBMANIFOLDS:

CHEKANOV, OH, M.