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Deformed Spaces and Symmetries

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Eur.Phys.J. C31 (2003) 129-138: M. Dimitrijević, L. Jonke, L. Möller, E. Tsouchnika, J. Wess, M. Wohlgenannt
Eur.Phys.J. C36 (2004) 117-126: M. Dimitrijević, F. Meyer, L. Möller, J. Wess

Why Noncommutative Geometry ?

- Divergences in QFT:
  Discrete space-time may lead to a finite or at least renormalisable theory (natural cutoff)

- Poor understanding of physics at short distances/high energies:
  Rich mathematical structures of deformed symmetries (Hopf algebras) give rise to new features

- Localization with extreme precision cause gravitational collapse → space-time below Planck scale has no operational meaning
  (→ K. Fredenhagen, S. Doplicher, J. E. Roberts)

- String Theory:
  Open strings in a magnetic background field
  → endpoints of open strings move on noncommutative D-branes
  (→ N. Seiberg, E. Witten)

- Loop Quantum Gravity:
  Discretization of space time in spin foam models

Noncommutative Spaces

- *Underlying idea:* Noncommutative Coordinates
  At very short distances: coordinates do not commute (Heisenberg 1930)

$$\boxed{[\hat{x}^i, \hat{x}^j] = C^{ij}(\hat{x}) \neq 0}$$

- differentiable space-time manifold $\longrightarrow$ algebra of noncommutative coordinates:

$$\widehat{\mathcal{A}}_{\hat{x}} = \mathbb{C}\langle\langle \hat{x}^1, \ldots, \hat{x}^n \rangle\rangle / ([\hat{x}^i, \hat{x}^j] - C^{ij}(\hat{x}))$$

- Good examples (Poincare Birkhoff Witt property):
  Freely generated algebras with the following commutation relations

1. canonical structure: $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$
2. Lie algebra structure: $[\hat{x}^i, \hat{x}^j] = iC^{ij}{}_k\hat{x}^k$
3. Quantum Space structure: $\hat{x}^i\hat{x}^j = q\hat{R}^{ij}{}_{kl}\hat{x}^k\hat{x}^l$

Symmetries

- ✘ In general background *Symmetries* are *broken* when passing over to noncommutative algebras:
  For example $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$ breaks Lorentz symmetry.

$$\Downarrow$$

- ✘ Symmetries have to be deformed, too:

Commutative Space + *Commutative* Symmetry	$\overset{\text{Deformation}}{\Longrightarrow}$	*Deformed* Space + *Deformed* Symmetry

- ✘ **Mathematics**: One can deform the algebra of functions on a Lie-group within the category of *Hopf algebras.*

- ✘ Deformed Spaces are (co-)modules of such deformed symmetries, i.e. we have a deformed symmetry acting on the deformed space

Canonically Deformed Spaces

- We note that

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \; \theta^{ij} = \mathrm{const}$$

breaks Lorentz-symmetry.

- So far no deformed Poincaré symmetry acting on this deformed space was known.

- Recently such a deformed symmetry was found by Chaichian et al. and independently in our group:

θ–DEFORMED POINCARÉ BIALGEBRA

$$\begin{aligned}
[\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0, \quad [\hat{\delta}_\omega, \hat{\partial}_\rho] = \omega_\rho{}^\mu \hat{\partial}_\mu, \\
[\hat{\delta}_\omega, \hat{\delta}'_\omega] &= \hat{\delta}_{\omega\times\omega'}, \quad (\omega\times\omega)'_\mu{}^\nu = -(\omega_\mu{}^\sigma \omega'_\sigma{}^\nu - \omega'_\mu{}^\sigma \omega_\sigma{}^\nu), \\
\Delta\hat{\partial}_\mu &= \hat{\partial}_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\partial}_\mu, \\
\Delta\hat{\delta}_\omega &= \hat{\delta}_\omega \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\delta}_\omega + \frac{i}{2}(\theta^{\mu\nu}\omega_\nu{}^\rho - \theta^{\rho\nu}\omega_\nu{}^\mu)\hat{\partial}_\rho \otimes \hat{\partial}_\mu
\end{aligned}$$

where $\hat{\delta}_\omega = -\hat{x}^\nu \omega_\nu{}^s \hat{\partial}_s + \frac{i}{2}\theta^{s\mu}\omega_s{}^\nu \hat{\partial}_\mu \hat{\partial}_\nu$

$\kappa-$Poincaré Algebra

- An example for the Lie-structure is the *$\kappa-$Deformed Spacetime:*

$$[\hat{x}^n, \hat{x}^j] = ia\hat{x}^j, \quad [\hat{x}^i, \hat{x}^j] = 0,$$

where $i, j = 0, 1, \ldots, n-1$ (in general: latin letters $\in \{0, 1, \ldots, n-1\}$, greek letters $\in \{0, 1, \ldots, n\}$)

- It is a module with respect to the *$\kappa-$Poincaré Algebra:* The algebra is undeformed

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M \\ [\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0\,, \end{aligned}$$

whereas the coalgebra is deformed

$$\begin{aligned} \Delta M^{ij} &= M^{ij} \otimes 1 + 1 \otimes M^{ij} \\ \Delta M^{in} &= M^{in} \otimes 1 + e^{ia\hat{\partial}_n} \otimes M^{in} + ia\hat{\partial}_k \otimes M^{ik} \\ \Delta \hat{\partial}_i &= \hat{\partial}_i \otimes 1 + e^{ia\hat{\partial}_n} \otimes \hat{\partial}_i \\ \Delta \hat{\partial}_n &= \hat{\partial}_n \otimes 1 + 1 \otimes \hat{\partial}_n\,. \end{aligned}$$

- Action on the coordinates:

$$[M^{ij}, \hat{x}^\mu] = \eta^{\mu j}\hat{x}^i - \eta^{\mu i}\hat{x}^j$$
$$[M^{in}, \hat{x}^\mu] = \eta^{\mu n}\hat{x}^i - \eta^{\mu i}\hat{x}^n + iaM^{i\mu}$$
$$[\hat{\partial}_i, \hat{x}^\mu] = \eta_i{}^\mu - ia\eta^{\mu n}\hat{\partial}_i, \ [\hat{\partial}_n, \hat{x}^\mu] = \eta_n{}^\mu \, .$$

- $\kappa-$*Poincaré Algebra* was first introduced by Lukierski et al.: Phys. Lett. **B264** (1991), Phys. Lett. **B293** (1992), ...

- *New* Motivation for considering $\kappa-$deformation of space-time: *Doubly Special Relativity*

  Maguejo and Smolin: Phys.Rev.Lett. **88** (2002);
  Amelino-Camelia et al. Class.Quant.Grav. **20** (2003);
  Lukierski and Nowicki: Int.J.Mod.Phys. **A18** (2003); ...

Derivatives

- *Derivatives* are maps on the deformed coordinate space:

$$\hat{\partial}^{\widehat{\mathcal{A}}} : \widehat{\mathcal{A}}_{\hat{x}} \to \widehat{\mathcal{A}}_{\hat{x}} \, .$$

  Thus, they have to be consistent with commutation relations of the coordinates.

- general ansatz:

$$[\hat{\partial}^{\widehat{\mathcal{A}}}_{\mu}, \hat{x}^{\nu}] = \delta^{\nu}_{\mu} + \sum_{j} A^{\nu \rho_1 \ldots \rho_j}_{\mu} \hat{\partial}_{\rho_1} \ldots \hat{\partial}_{\rho_j} \, .$$

- The huge freedom in the choice of derivatives can be reduced by requiring that the derivatives should be a module with respect to the deformed background symmetry

- κ–deformed case: derivatives $\hat{\partial}_i$ and $\hat{\partial}_n$ from above are obtained by requiring linearity in derivatives:

$$[\hat{\partial}_{\mu}, \hat{x}^{\nu}] \stackrel{!}{=} \delta^{\nu}_{\mu} + A^{\rho}_{\mu} \hat{\partial}_{\rho} \, .$$

✘ *of special interest*: look for derivatives $\hat{D}_\mu$ that transform like *vectors* with respect to κ−Poincaré:

$$[M^{\mu\nu}, \hat{D}_\mu] \stackrel{!}{=} \eta^\nu{}_\rho \hat{D}^\mu - \eta^\mu{}_\rho \hat{D}^\nu$$

✘ solution:

$$\begin{aligned} \hat{D}_n &= \frac{1}{a}\sin(a\hat{\partial}_n) - \frac{ia}{2}\hat{\partial}^l\hat{\partial}_l e^{-ia\hat{\partial}_n} \\ \hat{D}_i &= \hat{\partial}_i e^{-ia\hat{\partial}_n} \end{aligned}$$

acting in the following *non*-linear way on the coordinates:

$$[\hat{D}_n, \hat{x}^j] = -ia\hat{D}^j,$$

$$[\hat{D}_n, \hat{x}^n] = \sqrt{1 + a^2 \hat{D}^\mu \hat{D}_\mu},$$

$$[\hat{D}_i, \hat{x}^j] = \eta_i{}^j \left(-ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}^\mu \hat{D}_\mu}\right),$$

$$[\hat{D}_i, \hat{x}^n] = 0$$

Towards a Physical Theory

- ✘ To establish a physical theory we need to come from the *abstract algebra* to *complex numbers.*

- ✘ There are basically two ways how to proceed:

1. Study Representations of the NC algebra (cp. QM)

2. Star product approach and Seiberg-Witten map:

> Represent the noncommutative algebra on the algebra of commutative functions by a *star product* (next slide)
> and express noncommutative fields in terms of commutative ones by *Seiberg-Witten map* (next talk)

Star Products

- Vector space of formal power series in commutative coordinates is isomorphic to the vector space of formal power series in noncommutative coordinates

$$
\begin{aligned}
\rho : \mathbb{R}[[x^0, \ldots, x^n]] &\longrightarrow \quad \widehat{\mathcal{A}} \\
f(x^\mu) &\mapsto \quad \hat{f}(\hat{x}^\mu)
\end{aligned}
$$

- To transmit the noncommutativity to the algebra of commutative functions we define a *new product*, called **star product** by pulling back the product of the noncommutative algebra:

$$
\boxed{f(x^\mu) \star g(x^\mu) := \rho^{-1}(\hat{f}(\hat{x}^\mu) \cdot \hat{g}(\hat{x}^\mu))}
$$

- The star product is *not* unique since the isomorphism ρ (called *ordering prescription*) is not unique

- *Example:* Canonical Structure $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$
  Moyal-Weyl Product

$$
f \star g = \mu \circ e^{i\theta^{ij}\partial_i \otimes \partial_j}(f \otimes g) = fg + \frac{i}{2}\theta^{ij}(\partial_i f)(\partial_j g) + \ldots,
$$

where $\mu(f \otimes g) := fg$ is just the multiplication map.

- $\kappa-$*Deformed Spaces:* Symmetric ordered star product

$$
\begin{aligned}
f \star g(x) &= \lim_{\substack{z\to x\\ y\to x}} \exp\left(x^j \partial_{z^j} \left(\frac{\partial_n}{\partial_{z^n}} e^{-ia\partial_{y^n}} \frac{1-e^{-ia\partial_{z^n}}}{1-e^{-ia\partial_n}} - \right.\right.\\
&\left.\left. + x^j \partial_{y^j} \left(\frac{\partial_n}{\partial_{y^n}} \frac{1-e^{-ia\partial_{y^n}}}{1-e^{-ia\partial_n}} - 1 \right) \right) f(z)g(y) \\
&= f(x)g(x) + \frac{i}{2} C^{\mu\nu}_{\lambda} x^\lambda (\partial_\mu f)(\partial_\nu g) + \dots ,
\end{aligned}
$$

where

$$C^{\mu\nu}_{\lambda} = a(\eta^{\mu}{}_{n}\eta^{\nu}{}_{\lambda} - \eta^{\nu}{}_{n}\eta^{\mu}{}_{\lambda}) .$$

and

$$\partial_n := \partial_{z^n} + \partial_{y^n}$$

Star-Representation of Algebra Operators

- The differential operators acting on the noncommutative algebra operators can be represented by commutative functions and derivative acting on commutative functions

$$\begin{array}{ccc} \hat{f}(\hat{x}) & \xrightarrow{\hat{O}} & \hat{O}(\hat{f}(\hat{x})) \\ \rho^{-1}\downarrow & & \downarrow\rho^{-1} \\ f(x) & \xrightarrow{O^*} & O^*(f(x)) \end{array}$$

- Star representations of the derivatives $\hat{D}_\mu$:

$$\begin{aligned} D_n^* f(x) &= \left(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{ia\partial_n^2}\partial_j\partial^j\right) f \\ D_i^* f(x) &= \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\partial_i f(x) \end{aligned}$$

$$\begin{aligned} D_n^*(f(x)\star g(x)) = {} & (D_n^* f(x))\star(e^{-ia\partial_n}g(x)) \\ & +(e^{ia\partial_n}f(x))\star(D_n^* g(x)) \\ & -ia\left(D_j^* e^{ia\partial_n}f(x)\right)\star(D^{j*}g(x)), \end{aligned}$$

$$\begin{aligned} D_i^*(f(x)\star g(x)) &= (D_i^* f(x))\star(e^{-ia\partial_n}g(x)) \\ &\quad + f(x)\star(D_i^* g(x)) \end{aligned}$$

Summary

- Concept of deformed spaces:
  spacetime is *discrete* at very *short distances* resp. at *high energies*

- differentiable manifold $\Longrightarrow$ noncommutative algebra
  commutative product $\Longrightarrow$ star product
  Leibniz rule $\Longrightarrow$ deformed Leibniz rule
  (important for gauge theory via Seiberg-Witten map)

- We found a $\theta-$Deformed Poincaé Algebra as symmtery for the canonical case where $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$

- We introduced the $\kappa-$Deformed Space and the $\kappa-$Deformed Poincaré Algebra:
  Found derivatives $\hat{D}_\mu$ that transform like vectors

- We studied star product representations of the $\hat{D}_\mu$

Outlook

- Study implications of the new symmetry for canonically deformed spaces
- Formulate a gauge theory $\Rightarrow$ Marija's talk
- Study phenomenological consequences of noncommutativity

Noncommutative Gauge Theories $\Rightarrow$ Interesting concepts for a better understanding of physics at short distances