Characters of the Positive Energy UIRs of D=4 Conformal Super-symmetry

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PLAN

Introduction to characters

General character formulae

Specifics for $su(2,2/N)$

Decompositions

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Elementary introduction to characters

Take the algebra $sl(2)$ with generators $X^\pm, H$ and relations:

$$[X^+, X^-] = H, \qquad [H, X^\pm] = \pm 2X^\pm$$

Lowest weight modules are defined by a lowest weight $\Lambda \in \mathfrak{h}^*$ ($\mathfrak{h}$ spanned by H) and lowest weight vector v_0 so that

$$H v_0 = \Lambda(H) v_0$$

$$X^- v_0 = 0$$

Lowest weight Verma modules V^Λ are induced from 1-dim. repres of the Borel subalgebra $\mathcal{B}$ (spanned by H and X^-). In short

$$V^\Lambda \cong U(\mathfrak{g}^+) \otimes v_0 \qquad , \mathfrak{g}^+ \text{ spanned by } X^+$$

Simpler: it consists of vectors (omitting $\otimes$)

$$(X^+)^k v_0 \qquad , k = 0, 1, \ldots$$

Take now a formal variable t. We can represent the Verma module by the infinite series:

$$1 + t + t^2 + \cdots \quad = \frac{1}{1-t}$$

So we define the character of the Verma module V^Λ to be

$$\boxed{\mathrm{ch}\ V^\Lambda \equiv \frac{e(\Lambda)}{1-t}} \quad , t \equiv e(\alpha)$$

Actually, $e(\cdot)$ is a formal exponent on $\mathfrak{h}^*$ such that $e(0) = 1$, $e(\alpha)e(\alpha') = e(\alpha+\alpha')$
(though here we have only one root and also Λ is a multiple of α)

omit

How to obtain f.d. irreps from Verma modules?

→ From reducible Verma modules, this happens

whenever holds the BGG condition

$$(\rho-\Lambda,\beta) = \frac{m}{2}(\beta,\beta) \quad , \quad m\in\mathbb{N},\ \beta\in\Delta^+$$

$$\rho = \frac{1}{2}\sum_{\beta\in\Delta^+}\beta$$

in our case if $\Lambda = \frac{m+1}{2}\beta$

Then V^Λ contains a singular vector

$$v_1 = (X^+)^m v_0$$

s.t.
$$\begin{cases} H v_1 = \Lambda'(H) v_1, & \Lambda' = \Lambda + m\beta \\ X^- v_1 = 0 \end{cases}$$

in other language V^Λ contains a submodule

$$V^\Lambda \supset V^{\Lambda+m\beta}$$

In order to obtain an irred. representation we need to take the factor module

$$L_\Lambda = V^\Lambda / I^\Lambda \quad , \quad I^\Lambda \text{ is the image of } V^{\Lambda+m\beta}$$

i.e. L_Λ will consist of the vectors

$$X^k v_0 \quad , \quad k = 0,1,\dots,m-1$$

Obviously the character of L_Λ should (up to multiple)

$$1 + t + \cdots + t^{m-1}$$

more logically since L_Λ is a factor-module we should have

$$\mathrm{ch}\, L_\Lambda = \mathrm{ch}\, V^\Lambda - \mathrm{ch}\, V^{\Lambda + m\beta} =$$

$$= \frac{e(\Lambda)}{1-t} - \frac{e(\Lambda + m\beta)}{1-t} =$$

$$= \frac{e(\Lambda)(1 - e(m\beta))}{1-t} = \frac{e(\Lambda)(1-t^m)}{1-t}$$

$$= e(\Lambda)(1 + t + \cdots + t^{m-1})$$

Generalization: Let g any simple Lie algebra with root system Δ, positive roots Δ^+

The character of a Verma module V^Λ is

$$\mathrm{ch}\, V^\Lambda = e(\Lambda) \prod_{\alpha \in \Delta^+} \frac{1}{(1 - e(\alpha))}$$

finite dim. irreps are obtained when BGG holds for every simple root (and consequently for any other positive root)

In this case the character formula can be written compactly as

$$\mathrm{ch}\, L_\Lambda = \sum_{w \in W} (-1)^{\ell(w)} \,\mathrm{ch}\, V^{w \cdot \Lambda}$$

$W \to$ Weyl group

This is what we had for $sl(2)$, where

the Weyl group consists of 2 elements: identity 1 and s_β, taking into account

$$\Lambda + m\beta = s_\beta \cdot \Lambda$$

$\ell(w)$: length function: (minimal) # of simple reflections by which one can write w

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Supercase $\quad g$: basic classical superalgebra (excl. $osp(1/2n)$)

root system

$$\Delta^+ = \Delta_0^+ + \Delta_1^+ \qquad \text{even, odd}$$

$$\mathrm{ch}\, V^\Lambda = e(\Lambda) \left\{ \prod_{\alpha \in \Delta_0^+} \frac{1}{1 - e(\alpha)} \right\} \prod_{\alpha \in \Delta_1^+} (1 + e(\alpha))$$

explanation: odd elements are nilpotent, so only up to linear in all of them.

shorter: $\quad \mathrm{ch}\, V^\Lambda = \mathrm{ch}\, V_0^\Lambda \,\mathrm{ch}\, \hat{V}^\Lambda$

Suppose V^Λ is <u>not</u> reducible w.r.t. any odd root. Then the character formula is:

$$\mathrm{ch}\, L_\Lambda = \mathrm{ch}\, L^0_\Lambda \,\mathrm{ch}\, \hat{V}^\Lambda =$$

$$= \mathrm{ch}\, L^0_\Lambda \,\mathrm{ch}\, \hat{L}_\Lambda$$

Suppose now V^Λ is reducible w.r.t. odd root β, i.e. holds Kac condition

$$(\rho - \Lambda, \beta) = 0 \qquad , \quad (\beta, \beta) = 0$$

Then we have embedding

$$V^\Lambda \supset V^{\Lambda+\beta}$$

In analogy with the even case we define

odd reflection [VD + V. Petkova, 1985]

$$\hat{s}_\beta \cdot \Lambda = \Lambda + \beta , \qquad (\beta, \beta) = 0, \quad (\Lambda, \beta) \neq 0$$

Indeed the odd embedding above is provided by a singular vector

$$v_s^{\Lambda} = \mathcal{P}^{\beta} v_0 \quad \text{such that} \quad (\mathcal{P}^{\beta})^2 = 0$$

Factorization of the submodule I^{Λ} generated by v_s^{Λ} (I^{Λ} is smaller than $V^{\Lambda+\beta}$!) means that we eliminate the root vector X_{β}^{+} from the basis, i.e. the odd character part will be

$$\mathrm{ch}\, \hat{L}_{\Lambda} = \prod_{\substack{\alpha \in \Delta^{+} \\ \alpha \neq \beta}} (1 + e(\alpha))$$

This can be also written using the odd reflections:

using the factorization $\hat{L}_\Lambda = \hat{V}^\Lambda / \hat{I}^\Lambda$

we have $\quad ch\, \hat{L}_\Lambda = ch\, \hat{V}^\Lambda - ch\, \hat{I}^\Lambda$

We know that $\hat{I}^\Lambda$ is "smaller" than $\hat{V}^{\Lambda+\beta}$ – in characters it means:

NEZ

$$ch\, \hat{I}^\Lambda = \frac{1}{1+e(\beta)}\, ch\, \hat{V}^{\Lambda+\beta} = \frac{e(\beta)}{1+e(\beta)}\, ch\, \hat{V}^\Lambda$$

$$\Rightarrow \quad ch\, \hat{L}_\Lambda = ch\, \hat{V}^\Lambda - \frac{1}{1+e(\beta)}\, ch\, \hat{V}^{\hat{s}_\beta \cdot \Lambda}$$

$$= ch\, \hat{V}^\Lambda - \hat{s}_\beta \cdot ch\, \hat{V}^\Lambda =$$

$$= \sum_{\hat{s} \in \hat{W}_\beta} (-1)^{\ell(\hat{s})}\, \hat{s} \cdot ch\, \hat{V}^\Lambda$$

$\hat{W}_\beta = \{1, \hat{s}_\beta\}$ since higher actions give zero

Thus, we have an easy conjecture for the general character formula:

$$ch\, \hat{L}_\Lambda = \sum_{\hat{s} \in \hat{W}} (-1)^{\ell(\hat{s})}\, \hat{s} \cdot ch\, \hat{V}^\Lambda$$

where $\hat{W} = \prod_{\beta \in K} \hat{W}_\beta$

↳ the set for which holds Kac criterion for reducibility

But life is more complicated!

There are 2 obstacles:

1) descendant redundancies, i.e., elimination of odd roots for which Kac condition does not hold, and even worsely the opposite – non-elimination of such for which it holds!

2) redundancies, coming from the fact that we are looking for characters of a real form, and all states have to obey the condition of being physical.

Thus the general character formula may be written as:

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{\jmath}\in\hat{W}} (-1)^{\ell(\hat{\jmath})}\,\hat{\jmath}\cdot\left(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}\right)$$

counter-term

where $\hat{W} = \prod_{\beta\in K'} \hat{W}_\beta$

K' is the set of eliminated elements of the odd root lattice, i.e. roots and some non-roots (examples to be shown)

$\mathcal{R}$ represents elements which has to be taken out since they are not allowed also in the odd Verma module

→ Important: even without any reducibility, R may be non-zero!

Thus, we have to be more specific.

Our algebra is the Lie superalgebra

$$\mathcal{g} = su(2,2/N) = \mathcal{g}_0 \oplus \mathcal{g}_1 \qquad \text{(even + odd)}$$

with

$$\mathcal{g}_0 = su(2,2) \oplus u(1) \oplus su(N)$$

The physically relevant representations are labelled by the signatures

$$\chi = [d\,;\, j_1, j_2\,;\, z\,;\, r_1, \ldots, r_{N-1}]$$

where

d : conformal weight

j_1, j_2 : non-negative (half)integers labelling Lorentz f.d. irreps

z : represents $u(1)$

$r_1, \ldots, r_{N-1}$: non-negative integers, Dynkin labels of $su(N)$

There is rich interplay between $\mathcal{g}$ and $\mathcal{g}^{\mathbb{C}} = sl(4/N)$

Root system of $sl(4|N)$

α_{ij} , $1 \leq i < j \leq N+4$

even roots $\begin{cases} 1 \leq i<j \leq 4 & \text{(the 6 roots of } sl(4)\text{)} \\ 5 \leq i<j \leq 4+N & \text{the } \frac{N(N-1)}{2} \text{ roots of } su(N) \end{cases}$

$4N$ odd roots $1 \leq i \leq 4$, $5 \leq j \leq 4+N$

Reducibility of V^Λ , $\Lambda = \Lambda(\chi)$

$sl(4)$ ($su(2,2)$) : we calculate $n_{ij} = (\rho - \Lambda, \alpha_{ij}^\vee)$, $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$

simple roots $\begin{cases} n_{12} = 1 + 2j_1 \equiv n_1 \in \mathbb{N} \\ n_{23} = 1 - d - j_1 - j_2 \equiv n_2 \\ n_{34} = 1 + 2j_2 \equiv n_3 \in \mathbb{N} \end{cases}$

$n_{13} = 2 - d + j_1 - j_2 = n_1 + n_2$

$n_{24} = 2 - d - j_1 + j_2 = n_2 + n_3$

$n_{14} = 3 - d + j_1 + j_2 = n_1 + n_2 + n_3$

f.d. irreps $\equiv$ all $n_i \in \mathbb{N}$, but in all our cases we have:

$\boxed{n_1, n_3 \in \mathbb{N}, \quad n_2 \notin \mathbb{N}}$

omit

Odd reducibilities

4N roots, 4N possibilities, 4 series:

$$d = d^1_{NK} - z\,\delta_{N4} \qquad K = 1, .., N$$

$$d^1_{NK} \equiv 4 - 2K + 2j_2 + z + 2m_K - 2m/N$$

$$d = d^2_{NK} - z\,\delta_{N4}$$

$$d^2_{NK} \equiv 2 - 2K - 2j_2 + z + 2m_K - 2m/N$$

$$d = d^3_{NK} - z\,\delta_{N4}$$

$$d^3_{NK} \equiv 2 + 2K - 2N + 2j_1 - z - 2m_K + 2m/N$$

$$d = d^4_{NK} - z\,\delta_{N4}$$

$$d^4_{NK} \equiv 2K - 2N - 2j_1 - z - 2m_K + 2m/N$$

where

$$m_K \equiv \sum_{i=1}^{N-1} r_i$$ number of cells of the K-th row of the standard Young tableau

$$m \equiv \sum_{K=1}^{N-1} m_K = \sum_{K=1}^{N-1} K r_K$$ # number of cells

Note that for fixed module only <u>one</u> condition from each series can be satisfied. Thus at most 4 odd conditions may be satisfied.
Fortunately, for the UIR we shall require <u>only</u> coincidence of up to 2 odd reducibilities.

Unitarity

Theorem [V.D. + V. Petrova, 1985]

The complete list of lowest weight UIRs of $su(2,2/N)$ is:

$$d \geq d_{max} = \max(d^1_{N1}, d^3_{NN})$$

$$d = d^4_{NN} \geq d^1_{N1} \quad , \quad j_1 = 0$$

$$d = d^2_{N1} \geq d^3_{NN} \quad , \quad j_2 = 0$$

$$d = d^2_{N1} = d^4_{NN} \quad , \quad j_1 = j_2 = 0$$

where d_{max} is the threshold of the continuous unitary spectrum.

Note: in the last case we have

$$d = m_1, \quad z = \frac{2m}{N} - m_1$$

and for $\underline{N=1}$ by agreement $m_1 = m = 0$ (no Tableau for $su(1)$!), i.e. all quantum numbers are ~~missing~~ zero, i.e. this is the trivial 1-dim. UIR.

For $N=1$ the above list was given first by Flato and Fronsdal, 1984.

Even for $d > d_{max}$ things are not trivial for low values of spin and isospin, but the most interesting things happen at reducib. points.

Thus, only 4 odd reducib. points are playing a role, and they have partial ordering

$$d^1_{N1} > d^2_{N1} \quad , \qquad d^3_{NN} > d^4_{NN}$$

Thus, at most 2 such points may coincide. Thus, we have <u>single reducibility conditions</u> (SRC) and <u>double-reducibility-conditions</u> (DRC) situations

<u>Take first SRC cases (4 of them)</u>

$d = d_{max} = d^1_{N1} > d^3_{NN}$

$d = d^2_{N1} > d^3_{NN}, \quad j_2 = 0$

$d = d_{max} = d^3_{NN} > d^1_{N1}$

$d = d^4_{NN} > d^1_{N1}, \quad j_1 = 0$

even lat.

$\alpha_{3,4+N} = \gamma_4 + \gamma_2$

$\alpha_{4,4+N} = \gamma_4$

$\alpha_{15} = \gamma_3 + \gamma_1$

even lat.

$\alpha_{25} = \gamma_3$

Even singular vectors

$$v_j^1 = \left(X_1^+\right)^{1+2j_1} v_0$$

$$v_j^2 = \left(X_3^+\right)^{1+2j_2} v_0$$

$$v_j^j = \left(X_j^+\right)^{1+r_{N+4-j}} v_0 \qquad j = 5, \dots, N+3$$

odd singular vectors

Cartan elem.

$$v^1_{odd} = P_{3,4+N}\, v_0 = \left(X_4^+ X_2^+ (h_2 - 1) - X_2^+ X_4^+ h_2\right) v_0$$

$$v^2_{odd} = X_4^+ v_0$$

$$v^3_{odd} = P_{15}\, v_0 = \left(X_3^+ X_1^+ (h_1 - 1) - X_1^+ X_3^+ h_1\right) v_0$$

$$v^4_{odd} = X_3^+ v_0$$

exercise: check that polynom. are nilpotent (trivial for ex. 2,4)

We now introduce the factor module L_Λ.

Let us denote by $|0\rangle$ the lowest weight vector of L_Λ. Each singular vector of V^Λ exists, generates zero action on $|0\rangle$ (null)

Even

$$(X_1^+)^{1+2j_1}\ |0\rangle = 0$$

$$(X_2^+)^{1+2j_2}\ |0\rangle = 0$$

$$(X_j^+)^{1+r_{N+4-j}}\ |0\rangle = 0 \quad , \qquad j = 5, \dots, N+3$$

Odd whenever applicable

$$P_{3,4+N}|0\rangle = \left(2j_2\, X^+_{3\,4+N} - X_4^+ X_2^+\right)|0\rangle = 0$$

$$X_4^+|0\rangle = 0$$

$$P_{15}|0\rangle = \left(2j_1\, X_{15} - X_3^+ X_1^+\right)|0\rangle = 0$$

$$X_3^+|0\rangle = 0$$

We see that whenever the appropriate condition holds, then the corresponding odd root vector is annihilated, i.e.

$X_{3\,4+N}$ when $j_2 \neq 0$!

X_4^+

X_{15} when $j_1 \neq 0$!

X_3^+

Unexpected problems <u>Explanation</u>!

Thus, in these 2 cases, singular vectors and null action are descendant, but they are replaced by more subtle null action, not generated by the Kac root conditions:

$$X_3^+ X_1^+ X_3^+ |0\rangle = 0 \quad , \quad d = d^3_{NN}, \; j_1 = 0$$

$$X_4^+ X_2^+ X_4^+ |0\rangle = 0 \qquad d = d^1_{N1}, \; j_2 = 0$$

This is a peculiarity of the super case.

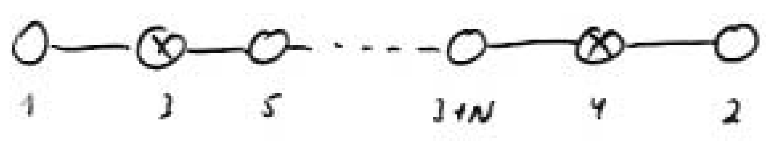
Thus, in these latter cases the character formula will be (the odd part of it)

$$\mathrm{ch}\, \hat{L}_\Lambda = \sum_{w \in W'} (-1)^w \, \hat{s}_w \cdot \mathrm{ch}\, \hat{V}^\Lambda \qquad (r_i \neq 0)$$

$$W' = \{1, s_{\alpha_1 + 2\alpha_3}\} \text{ or }$$

$$\{1, s_{\alpha_2 + 2\alpha_4}\}$$ ⇐ not roots !!

<u>More complications</u> if some $r_i = 0$. Here we need explicitly the Dynkin diagram:



This is a <u>non-distinguished</u> D.d. with 2 odd simple roots, most suitable for our purposes.

Example, $r_1 = 0$, $j_2 \neq 0$

Consider the vector

$$P_{3,3+N} v_0 = \left(2 j_2 X_{3,3+N} - X_{4,3+N} X_2^+\right) v_0 =$$

$$= \left(P_{3,4+N} X_{3+N}^+ - X_{3+N}^+ P_{3,4+N}\right) v_0$$

and we consider the case when $P_{3,4+N} |0\rangle = 0$

and since $X_{3+N}^+ |0\rangle = 0$ $\quad (\Leftarrow r_1 = 0)$

then $\Rightarrow$ $\quad P_{3,3+N} |0\rangle = 0$

i.e. for $j_2 \neq 0$ we eliminate also the root vector $X_{3,3+N}$

Analogously, if

$$r_1 = r_2 = \dots = r_{i_0} = 0$$

then all vectors $X_{3,4+N-i}$, $\quad i \leq i_0$

are eliminated; in particular,

if all $r_i = 0$, then <u>all</u> $X_{3,4+k}$ $\quad k = 1, \dots, N$ are eliminated.

The character f-le is

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{s \in \hat{W}^a_{i_0}} (-1)^{\ell(s)}\, s \cdot \mathrm{ch}\,\hat{V}^\Lambda$$

$$\hat{W}^a_{i_0} = \hat{W}_{\alpha_{3,N+4}} \times W_{\alpha_{3,N+3}} \times \dots \times \hat{W}_{\alpha_{3,N+4-i_0}}$$

Thus, we have illustrated peculiarities of the 1st kind, namely, elimination of vectors, that do not follow from Kac red. cond.

Now we turn to problems of second kind, namely, those that originate from the fact that we consider $su(2,2/N)$, not present for $sl(4/N)$.

This can be illustrated already for $N=1$. In that case we have 4 odd vectors. The claim is that sometimes they can not be applied on the vacuum to produce a component of a superfield.

The vacuum $|0\rangle$ carries the quantum numbers of the lowest weight vector v_0 i.e. for $N=1$

$$\chi = [d;\, j_1, j_2;\, z] \xrightarrow{\beta} \chi'$$

$$X^+_{\alpha_{3,N+4}} : \quad \chi' = [d+\tfrac{1}{2};\, j_1,\, \underbrace{j_2-\tfrac{1}{2}}_{};\, z+\tfrac{3}{2}]$$

$$X^+_{\alpha_{4,N+4}} : \quad \chi' = [d+\tfrac{1}{2};\, j_1,\, j_2+\tfrac{1}{2};\, z+\tfrac{3}{2})$$

$$X^+_{\alpha_{1,5}} : \quad \chi' = [d+\tfrac{1}{2};\, \underbrace{j_1-\tfrac{1}{2}}_{},\, j_2;\, z-\tfrac{3}{2}]$$

$$X^+_{\alpha_{25}} : \quad \chi' = [d+\tfrac{1}{2};\, j_1+\tfrac{1}{2},\, j_2;\, z-\tfrac{1}{2})$$

Obviously, if $j_2 = 0$, the state

$X^+_{\alpha_{3,N+1}} |0\rangle$ is not allowed

and if $j_1 = 0$, the state

$X^+_{15} |0\rangle$ is not allowed

For $N > 1$ there are similar restrictions from the values of r_i. Altogether, the general odd state looks as follows

$$\Psi = \left(\prod_{k=N}^{1} \left(X^+_{1,4+k} \right)^{\varepsilon_{1,4+k}} \right) \left(\prod_{k=N}^{1} \left(X^+_{2,4+k} \right)^{\varepsilon_{2,4+k}} \right) \times$$

$$\times \left(\prod_{k=1}^{N} \left(X^+_{3,4+k} \right)^{\varepsilon_{3,4+k}} \right) \left(\prod_{k=1}^{N} \left(X^+_{4,4+k} \right)^{\varepsilon_{4,4+k}} \right) |0\rangle$$

where $\varepsilon_{ij} = 0, 1$ indicates the presence of a gen. Conditions for a state to be allowed:

$$\sum_{k=1}^{N} \varepsilon_{1,4+k} \leq \sum_{k=1}^{N} \varepsilon_{2,4+k} + 2 j_1$$

$$\sum_{k=1}^{N} \varepsilon_{3,4+k} \leq \sum_{k=1}^{N} \varepsilon_{4,4+k} + 2 j_2 \qquad (*)$$

$$\varepsilon_{1,5+i} + \varepsilon_{2,5+i} + \varepsilon_{3,4+i} + \varepsilon_{4,4+i} \leq \varepsilon_{1,4+i} + \varepsilon_{2,4+i} + \varepsilon_{3,5+i} + \varepsilon_{4,5+i} + r_i$$

$$i = 1, \ldots, N-1$$

These conditions are also sufficient only for $N=1$. To formulate the exact condition we need the notion of level ε

$$\varepsilon = \sum_{k=1}^{N} \left(\varepsilon_{1,4+k} + \varepsilon_{2,4+k} + \varepsilon_{3,4+k} + \varepsilon_{4,4+k} \right)$$

i.e the total number of all generators applied on $|0\rangle$

Criterion: The necessary and sufficient conditions for a state of level ε to be allowed are that conditions $(*)$ are fulfilled and that the state is a descendant of an allowed state of level $\varepsilon - 1$.

Examples of states which do not fulfil the second part of the Criterion only, are given in the text.

Thus, even for UIRs which come from oddly irreducible Verma modules there will be lots of forbidden states at low spins and isospin

The maximal number of states is 2^{4N} and it is realized only if

$$j_1, j_2 \geq \frac{N}{2} \quad , \quad r_i \geq 4 \ , \ \forall i$$

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Decompositions

SRC:

$$\hat{L}_{long} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta}$$

as vector spaces

always for $N=1$
for $N>1$: some condition

$d = d_{max} = d^1 > d^3$, $r_1 > 0$

$j_2 > 0$, $\beta = \alpha_{3,4+N}$ (the root corresp. to d^1)

$j_2 = 0$, $\beta = \alpha_{3,4+N} + \alpha_{4+p,4+N}$ ($\leftarrow$ not a root!)

$d = d_{max} = d^3 > d^1$, $r_{N-1} > 0$

$j_1 > 0$, $\beta = \alpha_{15}$ (the root corresp. to d^3)

$j_1 = 0$, $\beta = \alpha_{15} + \alpha_{25}$ ($\leftarrow$ not a root!)

DRC:

$$\hat{L}_{long} = \hat{L}_\Lambda + \hat{L}_{\Lambda+\beta} + \hat{L}_{\Lambda+\beta'} + \hat{L}_{\Lambda+\beta+\beta'}$$

$r_1 r_{N-1} > 0$

$d = d_{max} = d^1 = d^3$

β, β' : 4 cases combinations of the above

UIRs that do not enter decompositions

SRC cases, $N > 1$

a) $d = d_{max} = d^1 > d^3$

$j_1, j_2 \sim \,, \quad r_1 = 0$

b) $d = d^2 > d^3$

$j_1 \sim, \; j_2 = 0, \quad r_1 \leq 2$

c) $d = d_{max} = d^3 > d^1$

$j_1, j_2 \sim \,, \quad r_{N-1} = 0$

d) $d = d^4 > d^1$

$j_1 = 0, \; j_2 \sim, \quad r_{N-1} \leq 2$

DRC cases

all cases when $N=1$

for $N>1$ the list is:

ac) $d = d_{max} = d^1 = d^3 = 2 + j_1 + j_2 + m_1$

$j_1, j_2 \sim, \quad r_1 r_{N-1} = 0$

ad) $d = d^1 = d^4 = 1 + j_2 + m_1$

$j_1 = 0, \ j_2 \sim, \ r_{N-1} \leq 2 \qquad \forall N > 1$

$r_1 = 0, \qquad N=2$

bc) $d = d^2 = d^3 = 1 + j_1 + m_1$

$j_1 \sim, \ j_2 = 0, \ r_1 \leq 2 \qquad \forall N > 1$

$r_{N-1} = 0, \qquad N=2$

bd) $d = d^2 = d^4 = m_1$

$j_1 = j_2 = 0,$

$r_1 \leq 4, \qquad N=2$

$r_1, r_{N-1} \leq 2, \qquad N > 2$