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Gauge Theory on κ-Minkowski Spacetime

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Reminder I

κ-Minkowski spacetime

coordinate algebra generated by $\hat{x}^\mu$, $\mu = 0, \dots n$ and

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\lambda \hat{x}^\lambda, \quad C^{\mu\nu}_\lambda = a(\eta^\mu{}_n \eta^\nu{}_\lambda - \eta^\nu{}_n \eta^\mu{}_\lambda),$$

$\eta^{\mu\nu} = diag(1, -1, \dots, -1)$ or

$$[\hat{x}^n, \hat{x}^j] = ia\hat{x}^j \;, \quad [\hat{x}^i, \hat{x}^j] = 0, \qquad (1)$$
$$i, j = 0, 1, \dots, n-1 \;, \quad a^n = a, \quad a^j = 0$$

symmetry generators

$$[M^{ij}, \hat{x}^\mu] = \eta^{\mu j}\hat{x}^i - \eta^{\mu i}\hat{x}^j,$$
$$[M^{in}, \hat{x}^\mu] = \eta^{\mu n}\hat{x}^i - \eta^{\mu i}\hat{x}^n + iaM^{i\mu} \qquad (2)$$

Leibniz rules for the symmerty generators

$$M^{ij}(\hat{f} \cdot \hat{g}) = (M^{ij}\hat{f}) \cdot \hat{g} + \hat{f} \cdot (M^{ij}\hat{g}),$$
$$M^{in}(\hat{f} \cdot \hat{g}) = (M^{in}\hat{f}) \cdot \hat{g} + (e^{ia\hat{\partial}_n}\hat{f}) \cdot (M^{in}\hat{g}) \qquad (3)$$
$$+ia(\hat{\partial}_k \hat{f}) \cdot (M^{ik}\hat{g})$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} \quad (4)$$

derivatives

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu + \sum_j A_\mu^{\nu\rho_1\ldots\rho_j} \hat{\partial}_{\rho_1} \ldots \hat{\partial}_{\rho_j} \tag{5}$$

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0 \tag{6}$$

additional requirement

$$[M^{\mu\nu}, \hat{D}_\rho] = \eta^\nu{}_\rho \hat{D}^\mu - \eta^\mu{}_\rho \hat{D}^\nu \tag{7}$$

$$\begin{aligned}
[\hat{D}_n, \hat{x}^j] &= -ia\hat{D}^j, \\
[\hat{D}_n, \hat{x}^n] &= \sqrt{1 + a^2 \hat{D}^\mu \hat{D}_\mu}, \\
[\hat{D}_i, \hat{x}^j] &= \eta_i{}^j \left(-ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}^\mu \hat{D}_\mu} \right), \\
[\hat{D}_i, \hat{x}^n] &= 0
\end{aligned} \tag{8}$$

$\star$-product

$$f \star g\,(x) = f(x)g(x) + \frac{i}{2}C^{\mu\nu}_{\lambda}x^{\lambda}(\partial_{\mu}f)(\partial_{\nu}g) + \dots \tag{9}$$

representation of the derivatives on commutative spacetime

$$\begin{aligned} D^*_n f(x) &= \Big(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{ia\partial_n^2}\partial_j\partial^j\Big)f(x), \\ D^*_i f(x) &= \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\partial_i f(x) \end{aligned} \tag{10}$$

Leibniz rules

$$\begin{aligned} D^*_n(f(x)\star g(x)) &= (D^*_n f(x))\star(e^{-ia\partial_n}g(x)) \\ &\quad +(e^{ia\partial_n}f(x))\star(D^*_n g(x)) \\ &\quad -ia\left(D^*_j e^{ia\partial_n}f(x)\right)\star(D^{j^*}g(x)), \\ D^*_i(f(x)\star g(x)) &= (D^*_i f(x))\star(e^{-ia\partial_n}g(x)) \\ &\quad +f(x)\star(D^*_i g(x)) \end{aligned} \tag{11}$$

Fields

$$\hat{\phi}'(\hat{x}') \to (1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\phi'(\hat{x})$$

because $\hat{x}'^{\mu}\hat{x}'^{\nu} \neq (1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{x}^{\mu}\hat{x}^{\nu}$

scalar field

$$(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{\phi}'(\hat{x}) = \hat{\phi}(\hat{x}) \tag{12}$$

vector field

$$(1+\epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{V}'_{\rho}(\hat{x}) = \hat{V}_{\rho}(\hat{x})+\epsilon_{\mu\nu}(\eta^{\nu}{}_{\rho}\hat{V}^{\mu}(\hat{x})-\eta^{\mu}{}_{\rho}\hat{V}^{\nu}(\hat{x})) \tag{13}$$

Reminder II

Classical (commutative) gauge theory

Gauge group, generators T^a, $[T^a, T^b] = i f^{ab}_{\ \ c} T^c$

$\psi^0(x)$ matter fields, $\delta_\alpha \psi^0(x) = i\alpha(x)\psi^0(x)$

$\alpha(x) = \alpha^a(x)T^a$, Lie algebra-valued gauge parameter

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi^0(x) = \delta_{-i[\alpha,\beta]}\psi^0(x)$$

$$\delta_\alpha(\partial_\mu\psi^0(x)) = \partial_\mu(i\alpha(x)\psi^0(x)) \neq i\alpha(x)\partial_\mu\psi^0(x)$$

Covariant derivative $\mathcal{D}_\mu\psi^0(x) = (\partial_\mu - iA^0_\mu(x))\psi^0(x)$

such that $\delta_\alpha(\mathcal{D}_\mu\psi^0(x)) = i\alpha(x)\mathcal{D}_\mu\psi^0(x)$

$A^0_\mu(x) = A^{0a}_\mu(x)T^a$, Lie algebra-valued gauge field

$$\delta_\alpha A^0_\mu = \partial_\mu\alpha - i[A^0_\mu, \alpha]$$

Field strenght $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu - i[A^0_\mu, A^0_\nu] = i[\mathcal{D}_\mu, \mathcal{D}_\nu]$

and $\delta_\alpha F^0_{\mu\nu} = i[\alpha, F^0_{\mu\nu}]$

$$S = S_{\text{matter}} + S_{\text{gauge}}$$

$S_{\text{gauge}} = -\frac{1}{4}\mathrm{Tr}\int \mathrm{d}^n x(F^0_{\mu\nu}F^{0\mu\nu}),\ S_{\text{matter}} = S_{\text{matter}}(\mathcal{D}_\mu\psi^0, \psi^0)$

1. Noncommutative gauge theory

$\star$-product formalisam

$\psi^0(x) \to \underline{\psi(x)}$ noncommutative field

$\alpha(x) \to \underline{\Lambda_\alpha(x)}$ noncommutative gauge parameter

$\delta_\alpha \psi^0(x) = i\alpha(x)\psi^0(x) \to \underline{\delta_\alpha \psi = i\Lambda_\alpha \star \psi(x)}$

$\mathcal{D}^0_\mu \psi^0(x) \to \underline{\mathcal{D}_\mu \psi(x)} = D^*_\mu \psi(x) - iV_\mu \star \psi(x)$

such that $\delta_\alpha(\mathcal{D}_\mu \psi(x)) = i\Lambda_\alpha \star \mathcal{D}_\mu \psi(x)$

$V_\mu(x)$ noncommutative gauge field

consistency check

$$\begin{aligned}(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\psi(x) &= (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi \\ &= \frac{1}{2}[\Lambda^a_\alpha \stackrel{\star}{,} \Lambda^b_\beta]\{T^a, T^b\} + \frac{1}{2}\{\Lambda^a_\alpha \stackrel{\star}{,} \Lambda^b_\beta\}[T^a, T^b] \quad (14)\end{aligned}$$

if $\Lambda_\alpha = \Lambda^a_\alpha T^a$, then (14) $\neq \delta_{-i[\Lambda_\alpha, \Lambda_\beta]}\psi(x)$

Two ways out

1) only $U(N)$ gauge theory alowed

2) enveloping algebra approach

2. Enveloping algebra approach

Basis in the enveloping algebra

$$
\begin{aligned}
:T^a: &= T^a, \\
:T^aT^b: &= \frac{1}{2}(T^aT^b + T^bT^a), \\
:T^{a_1}\dots T^{a_l}: &= \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\dots T^{\sigma(a_l)})
\end{aligned}
$$

$\Lambda_\alpha(x)$ enveloping algebra-valued

$$
\begin{aligned}
\Lambda_\alpha(x) &= \sum_{l=1}^{\infty}\sum_{\text{basis}} \alpha_l^{a_1\dots a_l}(x) :T^{a_1}\dots T^{a_l}: \\
&= \alpha^a(x) :T^a: + \alpha_2^{a_1a_2}(x) :T^{a_1}T^{a_2}: + \dots
\end{aligned}
$$

covariant derivative

$$
\begin{aligned}
\mathcal{D}_\mu &= D^*_\mu - iV_\mu, \\
V_\mu &= \sum_{l=1}^{\infty}\sum_{\text{basis}} V^l_{\mu\, a_1\dots a_l} :T^{a_1}\dots T^{a_l}:
\end{aligned}
$$

gauge field $V_\mu(x)$ also envelpoing algebra-valued

consistency condition (14) coloses but (apparently) infinitely many degrees of freedom

3. Seiberg-Witten map

solution in terms of Seiberg-Witten map

(degrees of freedom)$_{\text{comm.g.th.}}$ = (degrees of freedom)$_{\text{noncomm.g.th.}}$

express noncommutative gauge parameter (fields) in terms of commutative parameter and fields, $\Lambda_\alpha = \Lambda_\alpha(x; \alpha, A^0_\mu)$

then (14) becomes

$$(\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi + i(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) \star \psi = \delta_{-i[\alpha,\beta]} \psi \quad (15)$$

$$\Lambda_\alpha = \alpha + a\Lambda^1_\alpha + \ldots + a^k \Lambda^k_\alpha + \ldots$$

$$\delta_\alpha \Lambda^1_\beta - \delta_\beta \Lambda^1_\alpha - i[\alpha, \Lambda^1_\beta] - i[\Lambda^1_\alpha, \beta] - \Lambda^1_{-i[\alpha,\beta]} = -\frac{1}{2} x^\lambda C^{\mu\nu}_\lambda \{\partial_\mu \alpha, \partial_\nu \beta\}$$

$$\Lambda_\alpha = \alpha - \frac{1}{4} x^\lambda C^{\mu\nu}_\lambda \{A^0_\mu, \partial_\nu \alpha\} \quad (16)$$

solution (16) not unique up to the solutions of the homogeneous equation

freedom in Seiberg-Witten map. . .

matter fields

use $\delta_\alpha\psi = i\Lambda_\alpha \star \psi$, (16) and expansion $\psi = \psi^0 + a\psi^1 + \ldots$

$$\psi = \psi^0 - \frac{1}{2}x^\lambda C_\lambda^{\mu\nu} A_\mu^0 \partial_\nu \psi^0 + \frac{i}{8} x^\lambda C_\lambda^{\mu\nu} [A_\mu^0, A_\nu^0]\psi^0 \qquad (17)$$

gauge fields

$\mathcal{D}_\mu \psi = D_\mu^* \psi - iV_\mu \star \psi$

different basis of derivatives in κ-Minkowski spacetime $\Rightarrow$ different choices of covariant derivatives

prefered D_μ^*, since $[M^{\mu\nu}, D_\rho^*] = \eta_\rho^\nu D^{*\mu} - \eta_\rho^\mu D^{*\nu}$

and $\mathcal{D}_\mu = D_\mu^* - iV_\mu$ is κ-Lorentz covariant

standard way to proceed

$$\delta_\alpha(\mathcal{D}_\mu\psi) = i\Lambda_\alpha \star \mathcal{D}_\mu \psi$$

$$\ldots$$

$$\begin{aligned}(\delta_\alpha V_\mu) \star \psi &= D_\mu^*(\Lambda_\alpha \star \psi) - \Lambda_\alpha \star (D_\mu^* \psi) + i[\Lambda_\alpha \stackrel{\star}{,} V_\mu] \star \psi \\ &\neq (D_\mu^* \Lambda_\alpha) \star \psi + i[\Lambda_\alpha \stackrel{\star}{,} V_\mu] \star \psi\end{aligned}$$

nontrivial Leibniz rules for D_μ^* derivatives

1. $\mu = j$

$$(\delta_\alpha V_j) \star \psi = \underbrace{D_j^*(\Lambda_\alpha \star \psi)} - \Lambda_\alpha \star (D_j^* \psi) + i[\Lambda_\alpha \stackrel{\star}{,} V_j] \star \psi$$

$$(D_j^* \Lambda_\alpha) \star (e^{-ia\partial_n} \psi) + \Lambda_\alpha \star (D_j^* \psi)$$

$$= (D_j^* \Lambda_\alpha) \star e^{-ia\partial_n} \psi + i[\Lambda_\alpha \stackrel{\star}{,} V_\mu] \star \psi$$

derivative-valued gauge fields;

ansatz: $V_j \star \psi = A_j \star (e^{-ia\partial_n} \psi)$ or $V_j \star = A_j \star e^{-ia\partial_n}$

$$\delta_\alpha A_j = (D_j^* \Lambda_\alpha) + i \Lambda_\alpha \star A_j - i A_j \star \left(e^{-ia\partial_n} \Lambda_\alpha\right) \qquad (18)$$

used $e^{-ia\partial_n}(f \star g) = (e^{-ia\partial_n} f) \star (e^{-ia\partial_n} g)$ and omitted $e^{-ia\partial_n} \psi$

use (18), (16) and expansion $A_j = A_j^0 + a A_j^1 + \dots$

$$V_j = A_j^0 - ia A_j^0 \partial_n - \frac{ia}{2} \partial_n A_j^0 - \frac{a}{4} \{A_n^0, A_j^0\}$$
$$+ \frac{1}{4} x^\lambda C_\lambda^{\mu\nu} \Big(\{F_{\mu j}^0, A_\nu^0\} - \{A_\mu^0, \partial_\nu A_j^0\} \Big) \qquad (19)$$

2. $\mu = n$

$$\delta_\alpha V_n \star \psi = (D_n^* \Lambda_\alpha) \star e^{-ia\partial_n}\psi + \Big((e^{ia\partial_n} - 1)\Lambda_\alpha\Big) \star D_n^* \psi$$
$$-ia(D_j^* e^{ia\partial_n}\Lambda_\alpha) \star D^{*j}\psi + i[\Lambda_\alpha \stackrel{\star}{,} V_n] \star \psi$$

ansatz: $V_n \star \psi = V_{n1} \star (e^{-ia\partial_n}\psi) + V_{n2} \star (D_n^* \psi) + V_{n3}^l \star (D_l^* \psi)$

$$\begin{aligned} \delta_\alpha V_{n1} &= \ldots \\ \delta_\alpha V_{n2} &= \ldots \\ \delta_\alpha V_{n3} &= \ldots \end{aligned} \tag{20}$$

use (20), (16) and expansion in terms of a

$$V_n = A_n^0 - iaA^{0j}\partial_j - \frac{ia}{2}\partial_j A^{0j} - \frac{a}{2}A_j^0 A^{0j}$$
$$+\frac{1}{4}x^\lambda C_\lambda^{\mu\nu}\Big(\{F_{\mu n}^0, A_\nu^0\} - \{A_\mu^0, \partial_\nu A_n^0\}\Big) \tag{21}$$

field strength

$F^0_{\mu\nu} \rightarrow \mathcal{F}_{\mu\nu} = i[\mathcal{D}_\mu \stackrel{\star}{,} \mathcal{D}_\nu]$

V_μ derivative-valued $\Rightarrow \mathcal{F}_{\mu\nu}, \mathcal{F}_{\mu\nu} \star \mathcal{F}^{\mu\nu}$ devivative-valued

$\Rightarrow S_{\text{gauge}}$ derivative-valued?!

"gravity-like" solution

$$\begin{aligned} \mathcal{F}_{\mu\nu} = F_{\mu\nu} + T^\rho_{\mu\nu}\mathcal{D}_\rho + \ldots \\ + T^{\rho_1 \cdots \rho_l}_{\mu\nu} : \mathcal{D}_{\rho_1} \ldots \mathcal{D}_{\rho_l} : + \ldots \end{aligned} \tag{22}$$

$F_{\mu\nu}$ curvature-like term, to be used in the action

$T^\rho_{\mu\nu}, \ldots$ torsion-like terms, to be ignored in the action

Finally,

$$\mathcal{L} = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{gauge}}$$

$\mathcal{L}_{\text{gauge}} = c \,\text{Tr}\left(F^{\mu\nu} \star F_{\mu\nu}\right)$ and $\mathcal{L}_{\text{matter}} = \bar{\psi} \star (i\gamma^\mu \mathcal{D}_\mu - m)\,\psi$

$$\begin{aligned} \mathcal{L}_{\text{gauge}} = & -\frac{1}{4}\text{Tr} F^0_{\mu\nu} F^{0\mu\nu} \\ & - \frac{i}{8} x^\nu C^{\rho\sigma}_\nu \text{Tr}\left(\mathcal{D}^0_\rho F^{0\mu\nu} \mathcal{D}^0_\sigma F^0_{\mu\nu} + \frac{i}{2}\{A^0_\rho, (\partial_\sigma + \mathcal{D}^0_\sigma)(F^{0\mu\nu} F^0_{\mu\nu})\}\right. \\ & \left. - i\{F^{0\mu\nu}, \{F^0_{\mu\rho}, F^0_{\nu\sigma}\}\}\right) + \frac{ia}{4}\text{Tr}\left(\mathcal{D}^0_n (F^{0\mu\nu} F^0_{\mu\nu}) - \{\mathcal{D}^0_\mu F^{0\mu j}, F^0_{nj}\}\right) \end{aligned} \tag{23}$$

$$\begin{aligned} \mathcal{L}_{\text{matter}} = & \bar{\psi}^0 \left(i\gamma^\mu \mathcal{D}^0_\mu - m\right)\psi + \frac{i}{2} x^\nu C^{\rho\sigma}_\nu \overline{\mathcal{D}^0_\rho \psi^0}\; \mathcal{D}^0_\sigma \left(i\gamma^\mu \mathcal{D}^0_\mu - m\right)\psi^0 \\ & - \frac{i}{2} x^\nu C^{\rho\sigma}_\nu \bar{\psi}^0 \gamma^\mu F^0_{\mu\rho} \mathcal{D}^0_\sigma \psi^0 + \frac{a}{2}\bar{\psi}^0 \gamma^j \mathcal{D}^0_n \mathcal{D}^0_j \psi^0 + \frac{a}{2}\bar{\psi}^0 \gamma^n \mathcal{D}^0_j \mathcal{D}^{0j} \psi^0 \end{aligned} \tag{24}$$

4. Integral and the Action

$$\int : \hat{\mathcal{A}}_{\hat{x}} \to \mathbb{C}, \text{ such that}$$

$$\int (c_1 f + c_2 g) = c_1 \int f + c_2 \int g, \text{ linearity}$$

$$\int f \star g = \int g \star f, \text{ cyclicity (trace property)}$$

required by gauge invariance of the S_{gauge}; can be used to formulate variational principle

1st possibility: measure funcition

$$\int \mathrm{d}^{n+1} x\, \underline{\mu(x)}\, (f(x) \star g(x)) = \int \mathrm{d}^{n+1} x\, \underline{\mu(x)}\, (g(x) \star f(x))$$

$$\partial_n \mu(x) = 0, \qquad x^j \partial_j \mu(x) = -n\mu(x)$$

problems: solution is not unique, nonvanishing $a \to 0$ limit, explicit breaking of the κ-Poincaré symmetry

2nd possibility: quantum trace

variational principle, integral κ-Poincaré invariant

problem: does not lead to the cyclic integral $\Rightarrow$ no gauge invarant action

3rd possibility: ???

Summary and Outlook

- it is possible to construct gauge theories on κ-Minkowski spacetime

- gauge fields are enveloping algeba-valued, SW map has to be used to reduce degrees of freedom to classical ones

  gauge fields are derivative-valued, consequence of nontrivial Leibnic rules for κ-derivatives, "gravity-like" approach used

- an action invariant under κ-Poincaré **and** gauge symmetry open problem

- work on quantisation, deformed coservation laws,. . .