III Summer school in math. physics

Zlatibor, 2004

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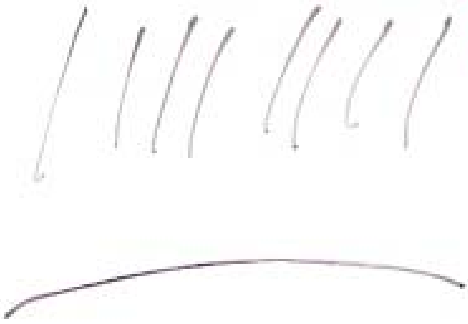
Self-duality, conformal geometry and curvature characterizations

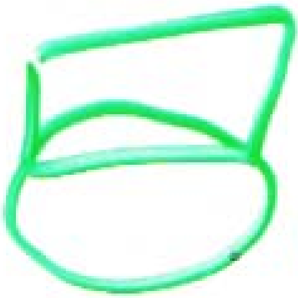
1. Notions and terminology
2. Self-duality
3. Symmetries and SD
4. ~~a) condo~~ Some characterizations and examples

1. Notions and terminology



E

$$\begin{array}{ccc} E & \leftarrow & G \\ \downarrow & & \\ M & & \end{array}$$



M

$$\begin{array}{c} TM \\ \downarrow \\ M \end{array} \qquad g, \nabla, R$$

$\tilde{g} = e^{2\varphi} g \quad \hat{W} = W$ Weyl

$\Lambda^* M$

$*$ - Hodge $*$ $\quad *: \Lambda^2 M \to \Lambda^2 M$

$*^2 = 1 \qquad (0,4), (2,2)$

$\Lambda^2 M = \Lambda^+ + \Lambda^-$

Terminology - Vocabular

total space of fibre bundle	P	space of phase factors
base space	M	space-time
structure group	G	gauge group
local sections of the principal bundles	f	local gauge
connection form on P	ω	gauge potencial
curvature form on P	Ω	gauge field
action of G in P $\psi_a : P \to P$	ψ_a	gauge transformation of the first kind

2. Self - duality

- Maxwell eq

E - el. field   B - magnetic

$\operatorname{div} B = 0 \qquad \operatorname{div} E = 0$

$\operatorname{curl} E + \frac{\partial B}{\partial t} = 0 \qquad \operatorname{curl} B - \frac{\partial E}{\partial t} = 0$

$F \in \Lambda^2 M$ (Faraday tensor)

$\downarrow$

$$dF = 0 \qquad d{*}F = 0$$

- curved space

$$YM : \mathcal{A} \to \mathbb{R}$$

$$YM(A) = \int_M F_A \wedge {*}F_A$$

- critical points

$$dF_A = 0 \quad , \quad d{*}F_A = 0$$

$$\boxed{{*}F_A = F_A}$$

- tangent bundle

$$*W = \pm W \qquad W = W_+ \oplus W_- \,, \quad W_\pm = 0$$

- metric signature

Riemannian $(++++)$

Kleinian $(--++)$

Lorentzian $(-+++)$ (cplx)

- Supersymmetry

$f : \Sigma \to X$ (4-dim) | standard σ-model admits $N=2$ sup. symm. extension

$\Updownarrow$

X is SD, E, scalar ≠ 0

- examples

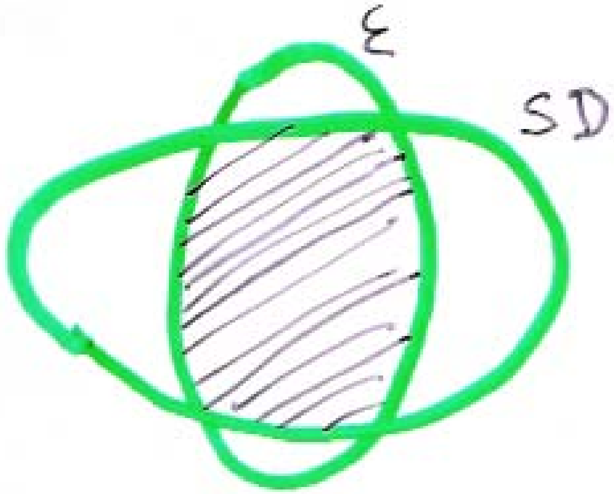
K3-surfaces

Eguchi-Hanson (hK)

Taub-NUT not symm.

3. Symmetries and SD

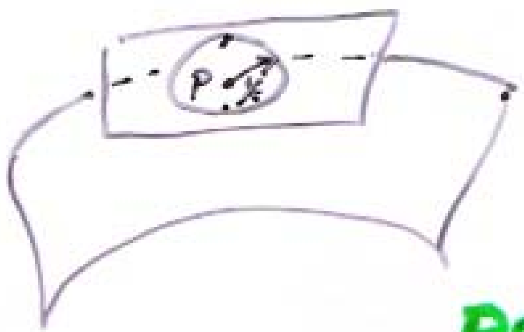
$\mathcal{E}$: $\quad R_{ij} = c\, g_{ij}$



- big moduli space

Biquard /0210059

($K + \mathcal{E} \longleftrightarrow SD + \mathcal{E}$)



Jacobi operator

$$\mathcal{J}_X(Y) = R(Y, X)X$$

Pointwise Osserman (pO)

$\Updownarrow$

eigenvalues of $\mathcal{J}_X$ are independent of X.

$SD + \mathcal{E} \longleftrightarrow pO \longleftrightarrow$ dynamically homog.

- $d=4$ $d=4k,\ k>1$

SD+E qK

$J_1J_2=-J_2J_1=J_3,\ J_1^2=J_2^2=-1,\ \nabla J_i=0$

- reductions

SD + symm $\longrightarrow$ SD, lower dim

- SD on $\mathbb{R}^{2,2}$ $\longrightarrow$ 2+1-chiral model (Ward)

Bäcklund transf.

4.1a

Conformally Osserman

P. Gilkey math/0311263,
IJGMMP, 2004

W – Weyl curvature tensor

$J(W)_X(Y) = W(Y,X)X$

Def Conformally Oss

– eigenvalues of $J(W)_X$ are independent on X

– Conformal invariant.

– Ex, $\mathbb{C}P^n$

Theorem Let (M,g) be a conformally Osserman manifold of dimension m.
If $m \equiv 2 \pmod 4 \geq 10$, then
M is conformally equivalent to an open subset of either $(\mathbb{C}P^k, g_{FS})$ or its noncompact dual.

4.b. SD + Lie group ?

$(++++)$ DeSmedt, Salamon, 2000

$[f_1, f_2] = f_2 - r f_3$ $\quad [f_2, f_3] = - f_4$

$[f_1, f_3] = r f_2 + f_3$

$[f_1, f_4] = 2 f_4$ $\quad$ not conf. flat

$(--++)$

S. Vukmirović

SD + Lie group + (0,2) Clif.

(G, g) $\quad J_1 J_2 = - J_2 J_1 = J_3$ $\quad J_i$ integrable

$J_1^2 = -1, \; J_2^2 = 1,$

(parahypercomplex) phC $\updownarrow$ SD

- Kamada - Machida, Dunajski, Ferapontov

- PHC6 - Sol_1^4-geometry Wall

$a = b = 0$

Inoue surface

Let us recall some standard notation of 4-dimensional Lie algebras (see [1]):

- $\mathbb{R} \oplus \mathfrak{sl}_2(\mathbb{R}):\ [e_1, e_2] = e_4,\ [e_2, e_4] = -e_1,\ [e_4, e_1] = e_2,$
- $\mathbb{R} \oplus \mathfrak{r}_{3,1}:\ [e_1, e_2] = e_2,\ [e_1, e_4] = e_4,$
- $\mathbb{R} \oplus \mathfrak{h}_3:\ [e_1, e_2] = e_3,$
- $\mathbb{R}^2 \oplus \mathfrak{aff}(\mathbb{R}):\ [e_1, e_2] = e_1,$
- $\mathbb{R} \oplus \mathfrak{d}_4:\ [e_1, e_4] = e_3,\ [e_1, e_2] = e_1,\ [e_2, e_4] = e_4,$
- $\mathfrak{d}_{4,\lambda}:\ [e_4, e_3] = e_3,\ [e_1, e_2] = e_3,\ [e_4, e_1] = \lambda e_1,\ [e_4, e_2] = (1-\lambda)e_2,$
- $\mathfrak{aff}(\mathbb{C}):\ [e_4, e_2] = e_2,\ [e_4, e_3] = e_3,\ [e_1, e_2] = e_3,\ [e_1, e_3] = e_2,$
- $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}):\ [e_1, e_3] = e_1,\ [e_2, e_4] = e_2,$
- $\mathfrak{t}_{4,1,\lambda}:\ [e_4, e_1] = e_1,\ [e_4, e_2] = e_2,\ [e_4, e_3] = \lambda e_3$, and
- $\mathfrak{h}_4:\ [e_4, e_3] = e_3,\ [e_1, e_2] = e_3,\ [e_4, e_2] = \frac{1}{2}e_2,\ [e_4, e_1] = e_2 + \frac{1}{2}e_1.$

In order to provide more uniform view we also use the following notation for 4 dimensional Lie algebras:

(PHC1) $\mathfrak{g}$ is abelian,
(PHC2) $[X, Y] = W,\ [Y, W] = -X,\ [W, X] = Y,$
(PHC3) $[X, Y] = Y,\ [X, W] = W,$
(PHC4) $[X, Y] = Z,$
(PHC5) $[X, Y] = X,$
(PHC6) $[X, W] = Z,\ [X, Y] = X,\ [Y, W] = W,$
(PHC7) $[X, Z] = X,\ [X, W] = Y,\ [Y, Z] = Y,\ [Y, W] = aX + bY,\ a, b \in \mathbb{R},$
(PHC8) $[X, Z] = X,\ [Y, W] = Y,$
(PHC9) $[Z, W] = Z,\ [Y, W] = Y,\ [X, W] = cX + aY + bZ,\ c \neq 0,\ a \in \mathbb{R},\ b \in \{0, 1\},$
PHC10) $[Y, X] = \lambda Z,\ [W, Z] = Z,\ [W, X] = \lambda X + bY + aZ,\ [W, Y] = (1-\lambda)Y,\ \lambda \neq$
0, 1